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ST. LAWRENCE HIGH SCHOOL

A Jesuit Christian Minority Institution

FOR GOD AND COUNTRY

Pre Annual Examination – 2020

Model answers

Sub: Mathematics | **Class: 11** | **F. M. : 80**

Duration: 3 hrs 15 mins | **Date: 20.01.2020**

Group – A

1. **Choose the correct option for the following questions** **(10x1=10)**

i) For two sets if $A \cup B = B \cap A$ then

a) $A \subseteq B$ b) $B \subseteq A$ c) $A = B$ d) none of these

Sol:c)

ii) Let A and B be two sets containing m and n distinct elements. Then number of relations from A to B is

a) 2^{m+n} b) 2^{n^m} c) 2^{m^n} d) 2^{mn}

Sol:d)

Iii $sec^2\theta = \frac{4xy}{(x+y)^2}$ is true iff

a) $x+y \neq 0$ b) $x = y, x \neq 0$ c) x=y d) $x,y \neq 0$

Sol: b)

iv) If –iy-x be a root of the equation $ap^2+bp+c=0$,then its another root will be

a) iy+x b) -iy+x c) iy-x d) none of these

Sol:c)

v) The number of permutations of n different things taken r at a time in which 4 particular things never occur is

a) ${}^{n}P_{r-4}$ b) ${}^{n-4}P_{r-4}$ c) ${}^{n-4}P_{r}$ d) ${}^{n}P_{r} - 4$

Sol:c)

vi) The coefficient of x^m in $(1+x)^{m+n}$ is

a) $\frac{m!n!}{(m+n)!}$ b) $\frac{(m+n)!}{m!n!}$ c) $(m+n)!$ d) $m!n!$

Sol:b)

vii) The eccentric angle of the point $(2,\frac{3\sqrt{3}}{2})$ on the ellipse $9x^2+16y^2=144$ is a)90^0 b)60^0 c)30^0 d)45^0

Sol:c)

viii)XOY- plane divides the join of (x,y,z) and (-y,-z,-x) in the ratio

a) x:z b) z:x c)y:z d)y:x

Sol:b)

ix)If y=$\frac{1}{\sin x\cos x}$,then $\frac{dy}{dx}$ is equal to

a)$\sec^2x$-cosec^2x b)$\text{cosec}^2x - \sec^2x$ c)cos2x d) sin2x

Sol:a)

x)The diameter of the circle concentric to the circle $x^2+y^2+4x-2y=20$ and passes through the origin is

a)10 units b) 4units c)$\sqrt{5}$ units d)none of these

Sol:c)

Group -B

2.a)Answer any two questions **2× 2 =4**

i) .Find the domain and range of f(x)= $\frac{x}{1+x^2}$

Sol:Domain=R,let $y=x/(1+x^2)$, $x^2y-x+y=0$,discriminant=$1-4y^2\geq 0, -1/2 \leq y \leq 1/2$ =range

ii)Define power set of a set A.Find the power set of A={{1},{2,3}}

Sol:The power set of a given set A is the set of all its subsets.P(A)={$\emptyset, \{\{1\}\}, \{\{2,3\}\}, A\}$

iii) Find the minimum value of $2^{sin^2\theta} + 2^{cos^2\theta}$

Sol:A.M≥G.M, $(2^{sin^2\theta} + 2^{cos^2\theta})/2 \geq (2^{sin^2\theta}2^{cos^2\theta})^{1/2}=\sqrt{2}$.. ans:$2\sqrt{2}$

iv)Prove $\frac{\sin(2A+B)}{\sin A} - 2\cos(A+B) = \frac{\sin B}{\sin A}$

Sol:$\frac{\sin\{(A+B)+A\}-2\cos(A+B)\sin A}{\sin A} = \frac{\sin(A+B)\cos B-\cos(A+B)\sin A}{\sin A} = \frac{\sin(A+B-A)}{\sin A}$

b) Answer any two questions **2× 2 =4**

i) .If the coefficients of x^r and x^{r+1} are equal in the expansion $(1+x)^{2n+1}$,then find r

Sol:$t_{r+1}={}^{2n+1}C_rx^r$,$t_{r+2}={}^{2n+1}C_{r+1}x^{r+1}$ so ${}^{2n+1}C_r={}^{2n+1}C_{r+1}$, r+r+1=2n+1, r=n

ii) Prove that $(0.444...)^{1/2}$= 0.666...

Sol: $(4/10+4/10^2+4/10^3+\ldots)^{1/2}=\left(\frac{\frac{4}{10}}{1-\frac{1}{10}}\right)^{1/2}=2/3=0.666\ldots$

iii) There are n points in a plane of which no three are collinear except m which are collinear. Find the number of straight lines and triangles formed.

Sol:straight lines=$^nC_2-{}^mC_2+1$,triangles= $^nC_3-{}^mC_3$

iv) Solve: $x^2-(3i-2\sqrt{3})x-6\sqrt{3}i=0$

Sol: $(x-3i)(x+2\sqrt{3})=0$, x=3i or $-2\sqrt{3}$

c) Answer any one question **2× 1 =2**

i) ii).The foci of the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ coincide with the hyperbola.If e=2 for hyperbola:find the equation for hyperbola.

Sol:e=$\sqrt{1-\frac{b^2}{a^2}}=1-\frac{9}{25}=\frac{4}{5}$ focus of ellipse=$(\pm ae,0)=(\pm 4,0)$=focus of hyperbola,a=2,b^2=12,so $\frac{x^2}{4}-\frac{y^2}{12}=1$

ii) Show that the points O(0,0,0),P(a,a,0),Q(a,0,a) and R(0,a,a) form a regular tetrahedron.

Sol:$OP^2=OQ^2=OR^2=PQ^2=QR^2=RP^2=2a^2$, $OP^2=(a-0)^2+(a-0)^2+0=2a^2$,4 faces are equilateral,OPQ..etc

d) Answer any one question **2× 1 =2**

i)$\lim_{x\to a}\frac{x\sin a-a\sin x}{x-a}$

Sol:let x-a=z,z tends to 0 when x tends to a

$\lim_{z\to 0}\frac{(z+a)\sin a-a\sin(a+z)}{z}=\sin a-a\lim_{z\to 0}\frac{2\cos\frac{z+2a}{2}\sin\frac{z}{2}}{z}=$
$\sin a-a\times\cos\frac{2a}{2}\times 1(\lim_{z\to 0}\frac{\sin z/2}{z/2}=1)$=sina-acosa

ii) If u and v are two differentiable functions of x,y=uv,prove that $\frac{y'}{y}=\frac{u'}{u}+\frac{v'}{v}$,where dash(′) denotes derivative w.r.t x

Sol: $y'=u'v+v'u$, product rule, $y'/y=(u'v+v'u)/uv$

e) Answer any one question **2× 1 = 2**

i) A coin is tossed is n times.Find the probability of odd number of heads.

Sol:$({}^nC_1+{}^nC_3+{}^nC_5+\ldots+)/2^n=2^{n-1}/2^n=1/2$

ii)Prove that the value of S.D does not depend on the origin of reference but depends on the unit of measurement(or scale of observation)

Sol:let x=a+by, $\overline{x} = a + b\overline{y}\,, \sigma_x^2 = \frac{\sum(x-\overline{x})^2}{n} = \frac{1}{n}\sum b^2(y-\overline{y})^2 = b^2\sigma_y^2$

GROUP-C

a) Answer any two questions: 4× 2=8

i) Prove that $4\sin 27^0 = \sqrt{5+\sqrt{5}} - \sqrt{3-\sqrt{5}}$

Sol: $(\sin27^0+\cos27^0)^2=1+2\sin27^0\cos27^0=1+\sin54^0=1+\cos36^0=1+\frac{\sqrt{5}+1}{4}$, $\sin27^0+\cos27^0=\frac{\sqrt{5+\sqrt{5}}}{2}$

$\sin27^0-\cos27^0=\frac{-\sqrt{3-\sqrt{5}}}{2}$ adding we get the result

ii)Prove $(A \cup B)^c = A^C \cap B^C$ by set algebra

Sol:let $x \in (A\cup B)^C$,$x \notin (A\cup B)$, $x\notin A$ OR $x\notin B$,$x\in A^C$ and $x\in B^C$, $x\in A^C\cap B^C$,so $(A\cup B)^C \subseteq A^C\cap B^C$...(1)

Let $y \in A^C\cap B^C$, $y \in A^C$ and $y \in B^C$,$y \notin A$ and $y \notin A\cup B$, $y \in (A\cup B)^C$,$A^C \cap B^C \sqsubseteq (A\cup B)^C$...(2).From (1),(2) result follows

iii) Solve $1 - 2\sin\theta - 2\cos\theta + \cot\theta = 0 (0 < \theta < 2\pi)$

Sol:$\sin\theta - 2\sin^2\theta - 2\cos\theta\sin\theta + \cos\theta = 0, (1-2\sin\theta)(\sin\theta+\cos\theta) = 0, \tan\theta = -1, \theta = \frac{3\pi}{4}, \frac{7\pi}{4}, \sin\theta = \frac{1}{2}, \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

b) Answer any two questions: 4× 2=8

i) If x,y,b are real, z=x+iy and $\frac{z-i}{z-1}$=ib, show that $(x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 = \frac{1}{2}$

Sol:$\frac{x+iy-i}{x+iy-1} = ib, \frac{(x+iy-i)(x-1-iy)}{(x-1+iy)(x-1-iy)} = ib, \frac{x(x-1)-y(1-y)}{(x-1)^2+y^2} = 0, x^2+y^2-x-y = 0$

ii) Prove by principle of mathematical induction, that $n(n^2-1)$ is divisible by 24,n is an odd positive integer

Sol:P(1)=0 divisible by 24,let P(n)=2k-1,so $n^2(n-1)=4k(k-1)(2k-1)$.Let P(m) be true,so $4m(m-1)(2m-1)=24r$,or $2m^3=6r+3m^2-m$.$P(m+1)=4(m+1)m(2m+1)=4(2m^3+3m^2+m)=4(6r+3m^2-m+3m^2+m)=24(r+m)^2$ which is divisible by 24

iii) m men and n women are to be seated in a row so that no two women sit together. If $m > n$, show that the number of ways in which they can be seated is $\frac{m!(m+1)!}{(m-n+1)!}$

Sol:m number of men can sit in m places in m! ways.n number of females can be seated in between m men and two extreme positions in(m-2+1)=m+1 places.so n number of females can occupy n places in ${}^{m+1}P_n$ ways.Ans: m! × ${}^{m+1}P_n$

iv)If the term independent of x in the expansion of $(\frac{k}{3}x^2 - \frac{3}{2x})^9$ be 2268,find k

Sol:$t_{r+1}={}^9C_r(\frac{kx^2}{3})^{9-r}(\frac{-3}{2x})^r$ 18-3r=0,r=6,so 7^{th} term is independent of x,$t_7=\frac{9.8.7}{3.2}k^3\frac{3^3}{2^6} = 2268, k = 4$

v) How many words can be made from the letters of the word COSTING so that

a) the vowels are always together,

b) may occupy only odd positions

Sol:a)six letters can be arranged in 6!=720 ways and two vowels can be arranged in 2! ways.number of words 720.2=1440

b)There are 4 odd places(1st,3^{rd},5^{th},7^{th}).So two vowels can be arranged in ${}^4P_2$=12ways.5 consonants can be arranged in 5! Ways.so total=12.120=1440

c) Answer any two questions **4× 2=8**

i) A ray of light starting from P(1,2), reflects on x-axis at A and hence passes through Q(5,3),find the coordinates of A

Sol:Let the point be (x,0).$\tan\theta = \frac{3-0}{5-x}, \tan(\pi-\theta) = \frac{2-0}{1-x}, \frac{-3}{5-x} = \frac{2}{1-x}, x = \frac{13}{5}$ so(13/5,0)

ii) If the circles $x^2+y^2+2ax+c^2=0$ and $x^2+y^2+2by+c^2=0$ touch each other,prove that $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}$

Sol: $r_1=\sqrt{a^2-c^2}, r_2 = \sqrt{b^2-c^2}, c_1 = (-a,0), c_2 = (0,-b), so c_1c_2 = r_1 \pm r_2$,for internally and externally touching,$\sqrt{a^2+b^2} = \sqrt{a^2-c^2} \pm \sqrt{b^2-c^2}$,squaring two times we get the result

iii) The equations of two adjacent sides of a parallelogram are 4x+5y=0 and 7x+2y=0.If the equation of its one diagonal be 11x+7y=9,find equation of its other diagonal.

Sol:7x+2y=0...(1) 4x+5y=0...(2) diagonal passes through origin,solving (1) & (3),A(-2/3,7/3).From (2) & (3), C(5/3,-4/3). Midpoint of AC IS O(1/2,1/2).Equation of BD,$y=\frac{\frac{1}{2}-0}{\frac{1}{2}-0}x$,x=y

d) Answer any one questions: **4× 1=4**

i) Evaluate.$\lim_{x\to1}\frac{\sum_{i=1}^{n}x^i-n}{x-1}$

Sol: $\lim_{x\to 1}\frac{x+x^2+x^3+\cdots+x^n-n}{x-1}=\lim_{x\to 1}\frac{x-1+x^2-1+x^3-1+\cdots+x^n-1}{x-1}=\lim_{x\to 1}(1+2x^{2-1}+3x^{3-1}+\cdots+nx^{n-1})=1+2+3+\cdots+n=n(n+1)/2$

ii) Find from first principle, the derivative of $\sqrt{\tan x}$

Sol: $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{\tan(x+h)}-\sqrt{\tan x}}{h}=\lim_{k\to 0}\frac{\sqrt{u+k}-\sqrt{u}}{k}\lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h}, u=\tan x, k=\tan(x+h)-\tan x, as\ h\to 0, k\to 0, =\lim_{k\to 0}\frac{k}{k(\sqrt{u+k}+\sqrt{u})}\lim_{h\to 0}\frac{sinh}{h}\lim_{h\to 0}\frac{1}{cosxcos(x+h)}=$
$\frac{1}{2\sqrt{u}}\sec^2 x=\sec^2 x/2\sqrt{tanx}$

e) Answer any one question: **4× 1=4**

i)Prove that $\log_2 3$ is irrational.(Use method of contradiction)

Sol: let $\log_2 3=\frac{p}{q}$, $2^{\frac{p}{q}}=3$, $2^p=3^q$ l.h.s is even and r.h.s is odd, so there is a contradiction.

ii) Check the validity of the following biconditional statement:

"If the two natural numbers x and y are odd, if and only if xy is an odd number"

Sol:Let p: x& y are odd, let q:xy is odd ,Let p be true,so x,y are odd implies x=2m+1,y=2n+1, m,n are natural numbers,xy=(2m+1)(2n+1)=2t+1.i.e odd number therefore p implies q, xy=2p+1,an odd number so xy=(2m+1)(2n+1)=product of two odd numbers therefore q implies p so p$\Leftrightarrow q$

f) Answer any one question: **4× 1=4**

i)Show that the probability that exactly one of the events A and B occurs is P(A)+P(B)-2P(AB)

Sol: $P[(A^C\cap B)\cup(B^C\cap A)]=P(A^C\cap B)+P(B^C\cap A)=P(B)-P(A\cap B)+P(A)-P(A\cap B)$

ii)The means of two sample sizes 500 and 600 were respectively 186 and 175.The corresponding standard deviations were respectively 9 and 10.The variable studied was height in cm. Obtain the mean and variance of the combined sample.

Sol: $\overline{x}=(n_1\overline{x_1}+n_2\overline{x_2})/n_1+n_2=\frac{500\times186+600\times175}{500+600}=180$, $(n_1+n_2)\sigma^2=n_1\sigma_1^2+n_2\sigma_2^2+n_1d_1^2+n_2d_2^2, (500+600)\sigma^2=500\times 9^2+600\times 10^2+500((186-180)^2+600(175-180)^2)), \sigma^2=\frac{1335}{11}=121.36\text{cm}^2$

GROUP-D

a) Answer any one question **5× 1=5**

i) If $\frac{\sin A}{\sin C}=\frac{\sin(A-B)}{Sin(B-C)}$,then show that a^2,b^2,c^2 are in A.P

Sol:sinA/sinC=(sinAcosB-cosAsinB)/(sinBcosC-cosBsinC), a/c=[a.(a^2+c^2-b^2)/2ac – b.(c^2+b^2-a^2)/2bc]/[b.(a^2+b^2-c^2)/2ab – c.(a^2+c^2-b^2)/2ac], b^2-c^2=a^2-b^2

ii) The two different values of θ,namely $\theta_1,\theta_2(0\le\theta_1,\theta_2\le 2\pi)$ satisfy the equation $\sin(\theta+\varphi)=\frac{1}{2}\sin 2\varphi$.Prove that $\frac{\sin\theta_1+\sin\theta_2}{\cos\theta_1+\cos\theta_2}=\cot\varphi$

Sol: $\sin(\theta_1+\varphi)=\frac{1}{2}\sin 2\varphi\,,\sin(\theta_2+\varphi)=\frac{1}{2}\sin 2\varphi,\theta_1+\varphi=\pi-\theta_2-\varphi,\theta_1+\theta_2=\pi-2\varphi,$

$$l.h.s=\frac{2\sin\frac{\theta_1+\theta_2}{2}\cos\frac{\theta_1-\theta_2}{2}}{2\cos\frac{\theta_1+\theta_2}{2}\cos\frac{\theta_1-\theta_2}{2}}=\tan\frac{\theta_1+\theta_2}{2}=\cot\varphi$$

b) Answer any two questions **$5\times 2=10$**

i) A polygon has 25 sides,the lengths of which starting from the smallest sides are in A.P.If the perimeter of the polygon is 1100cm and the length of the largest side is 10 times that of the smallest,find the length of the smallest side and the common difference of the A.P

Sol:let the sides be a,a+d,...,a+24d.by problem 25/2{2a+24d}=1100 or a+12d=44...(1) 10a=a+24d or 3a=8d...(2).Solving a=8,d=3

ii) Show that the value of the middle term in the expansion of $(x+\frac{1}{2x})^{2n}$ is $\frac{1.3.5...(2n-1)}{n!}$

Sol:There is one middle term as it contains 2n+1 odd terms i.e 2n/2 +1 i.e (n+1)th term=t_{n+1}=${}^{2n}C_n x^{2n-n}$.$(1/2x)^n$= (2n)!/(n!n!)$\times x^n\frac{1}{2^n x^n}=\frac{[2n.(2n-2)...6.4.2][(2n-1)...5.3.1]}{2^n n!n!}=\frac{2^n[n(n-1)...3.2.1][(2n-1)...5.3.1]}{2^n n!n!}$

iii) Solve|x-1|+|x-2|+|x-3|≥3, x is real number

Sol:-(x-1)-(x-2)-(x-3)$\ge 3, when\ x\le 1$,so x$\le 1:$,x-1+x-2-(x-3)$\ge 3, when\ 2\le x\le 3, so\ x\ge 3\ which\ does\ not\ belong\ in[2,3], x-1+x-2+x-3\ge 3, when\ x\ge 3$ so x$\ge$ 3

iv) Find the sum to n terms: 2+3.3+4.3^2+5.3^3+...

Sol:let S be the sum. -2S= S-3S = 2+{3+3^2+...+(n-1)terms}=2+3.$\frac{3^{n-1}-1}{3-1}$ –(n+1)3^n, S=1/4{3^n(2n+1)-1}

c) Answer any one question **5× 1=5**

i)Prove that $S_2P+S_1P=20$ for the ellipse $\frac{x^2}{100}+\frac{y^2}{36}=1$,$S_2$ and S_1 are the two foci of the ellipse and P is any point on the ellipse.

Sol:a=10,b=6,let P($10\cos\theta, 6\sin\theta$) be any point on ellipse. e=4/5 as e=$(1-b^2/a^2)^{1/2}$,coordinates of foci=$(\pm ae, 0) = (\pm 8,0)$.distance of P from foci is $\sqrt{(10\cos\theta - 8)^2 + (6\sin\theta)^2} + \sqrt{(10\cos\theta + 8)^2 + (6\sin\theta)^2} = (10 - 8\cos\theta) + (10 + 8\cos\theta) = 20$

ii) The directrix of parabola is $x + y + 4 = 0$ and vertex is the point (-1,-1).Find the coordinates of focus and equation of parabola.

Sol:Since axis of parabola is perpendicular to the directrix and passes through (-1,-1),so equation of directrix is y-(-1)=1[x-(-1)] i.e.x-y=0.point of intersection of axis and directrix is found by solving x+y+4=0 and x-y=0 i.e x=-2,y=-2.let coordinates of focus be(h,k). so -1=(-2+h)/2,-1=(-2+k)/2, so (h,k)=(0,0).SP=PM implies $(x^2+y^2)=(x+y+4)^2/2$.symbols have usual meaning.