ST. LAWRENCE HIGH SCHOOL

A JESUIT CHRISTIAN MINORITY INSTITUTION

FOR GOD AND COUNTRY

STUDY MATERIAL-4

SUBJECT – MATHEMATICS

1st term

Chapter: Trigonometry **Class: XI**

Topic: Trigonometric Identities **Date: 20.06.2020**

Trigonometric Ratios and Identities (Theory) :-

Trigonometric Ratios and Identities

2.1 Introduction

Trigonometry is a branch of Mathematics that relates to the study of angles, measurement of angles and units of measurement. It also concerns itself with the six ratios for a given angle and the relations satisfied by these ratios.

In an extended way, it is also a study of the angles forming the elements of a triangle. Logically, a discussion of the properties of a triangle, solving problems related to triangles, physical problems in the area of heights and distances using the properties of a triangle – all constitute a part of the study. It also provides a method of solution of trigonometric equations.

2.2 Definitions

1. **Angle:** The motion of any revolving line in a plane from its initial position (initial side) to the final position (terminal side) is called angle (Fig. 2.1). The end point O about which the line rotates is called the **vertex of the angle**.

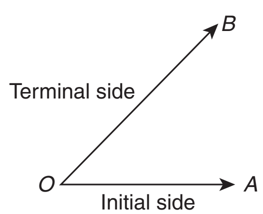

Figure 2.1

2. **Measure of an angle:** The measure of an angle is the amount of rotation from the initial side to the terminal side.
3. **Sense of an angle:** The sense of an angle is determined by the direction of rotation of the initial side into the terminal side. The sense of an angle is said to be positive or negative according to the rotation of the initial side in anticlockwise or clockwise direction to get to the terminal side (Fig. 2.2).

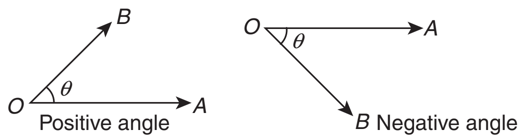

Figure 2.2

4. **Right angle:** If the revolving ray, starting from its initial position to final position, describes one quarter of a circle, then we say that the measure of the angle formed is a right angle.
5. **Quadrants:** Let *X′OX* and *YOY′* be two lines at right angles in the plane of the paper (Fig. 2.3). These lines divide the plane of paper into four equal parts known as quadrants. The lines *X′OX* and *YOY′* are known as *x*-axis and *y*-axis, respectively. These two lines taken together are known as **coordinate axes**.

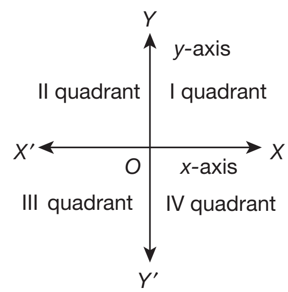

Figure 2.3

6. **Angle in standard position:** An angle is said to be in standard position if its vertex coincides with the origin *O* and the initial side coincides with *OX*, that is, the positive direction of *x*-axis.
7. **Angle in a quadrant:** An angle is said to be in a particular quadrant if the terminal side of the angle in standard position lies in that quadrant.
8. **Quadrant angle:** An angle in standard position is said to be a quadrant angle if the terminal side coincides with one of the axes.

2.3 Measurement of Angles

There are three systems for measuring angles.

1. **Sexagesimal or English system:** Here a right angle is divided into 90 equal parts known as **degrees**. Each degree is divided into 60 equal parts called **minutes** and each minute is further divided into 60 equal parts called **seconds**. Therefore,

$$1 \text{ right angle} = 90 \text{ degree } (= 90^\circ)$$
$$1^\circ = 60 \text{ min } (= 60')$$
$$1' = 60 \text{ s } (= 60'')$$

2. **Centesimal or French system:** It is also known as French system. Here a right angle is divided into 100 equal parts called **grades** and each grade is divided into 100 equal parts called **minutes** and each minute is further divided into 100 equal parts called **seconds**. Therefore,

$$1 \text{ right angle} = 100 \text{ grades} = (100^g)$$

1 grade = 100 min (= 100′)
1 min = 100 s (= 100″)

3. **Circular system:** In this system, the unit of measurement is **radian**. One radian, written as 1^c, is the measure of an angle subtended at the centre of a circle by an arc of length equal to the radius of the circle.

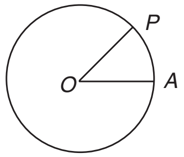

Figure 2.4

Consider a circle of radius r having centre at O (Fig. 2.4). Let A be a point on the circle. Now cut off an arc AP whose length is equal to the radius r of the circle. Then by definition the measure of $\angle AOP$ is 1 radian ($= 1^c$).

2.4 Relation Between Three Systems of Measurement and Angle

Let D be the number of degrees, R be the number of radians and G be the number grades in an angle θ. Now

$$90° = 1 \text{ right angle} \Rightarrow 1° = \frac{1}{90} \text{ right angle}$$

$$\Rightarrow D° = \frac{D}{90} \text{ right angles} \Rightarrow \theta = \frac{D}{90} \text{ right angles} \tag{2.1}$$

Again, π radians = 2 right angles $\Rightarrow$ 1 radian $= \dfrac{2}{\pi}$ right angles

$$\Rightarrow R \text{ radians} = \frac{2R}{\pi} \text{ right angles} \Rightarrow \theta = \frac{2R}{\pi} \text{ right angles} \tag{2.2}$$

And 100 grades = 1 right angle $\Rightarrow$ 1 grade $= \dfrac{1}{100}$ right angle

$$\Rightarrow G \text{ grades} = \frac{G}{100} \text{ right angles} \Rightarrow \theta = \frac{G}{100} \text{ right angles} \tag{2.3}$$

From Eqs. (2.1)–(2.3), we get

$$\frac{D}{90} = \frac{G}{100} = \frac{2R}{\pi}$$

This is the required relation between the three systems of measurement of an angle.

2.5 Relation Between Arc and Angle

If s is the length of an arc of a circle of radius r, then the angle θ (in radians) subtended by this arc at the centre of the circle (Fig. 2.5) is given by

$$\theta = \frac{s}{r} \text{ or } s = r\theta$$

$$\text{Arc} = \text{Radius} \times \text{Angle in radians}$$

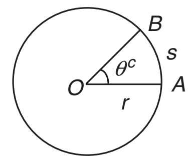

Figure 2.5

Sectorial area: Let OAB be a sector having central angle θ^c and radius r. Then area of the sector OAB is given by

$$\frac{1}{2} r^2 \theta^c$$

Note: π is a real number whereas π^c stands for 180°.
Remember the relation,

$$\pi \text{ radians} = 180° = 200^g$$

$$1 \text{ radian} = \frac{2}{\pi} \times \text{right angle} = \frac{180°}{\pi}$$

$$= 180° \times 0.3183098862\ldots = 57.2957795°$$

$$= 57°17'44.8'' \text{ (nearly)}$$

Illustration 2.1 Find the radian measure corresponding to $-37°\ 30'$.

Solution:

We know that $60' = 1°$. Therefore

$$30' = \left(\frac{1}{2}\right)^{\circ};\ -37°\ 30' = -\left(37\frac{1}{2}\right)^{\circ} = -\left(\frac{75}{2}\right)^{\circ}$$

As $360° = 2\pi$ radians, we have

$$-\left(\frac{75}{2}\right)\frac{\pi}{180} \text{ radians} = -\frac{5\pi}{24} \text{ radians}$$

Illustration 2.2 The minute hand of a clock is 10-cm long. How far does the tip of the hand move in 20 min?

Solution:

The minute hand moves through 120° in 20 min or moves through $2\pi/3$ radians. Since the length of the minute hand is 10 cm, the distance moved by the tip of the hand is given by the formula

$$l = r\,\theta = 10 \cdot \frac{2\pi}{3} = \frac{20\pi}{3} \text{ cm}$$

Illustration 2.3 A rail road curve is to be laid out on a circle. What radius should be used if the track is to change direction by 25° in a distance of 40 meters?

Solution:

The angle in radian measure $= \dfrac{25\pi}{180} = \dfrac{5\pi}{36}$

If r is the radius of the circle, using $l = r\,\theta$, we have

$$r = \frac{l}{\theta} = \frac{40}{\frac{5\pi}{36}} = \frac{288}{\pi} = 91.636 \text{ m}$$

Illustration 2.4 The circular wire of radius 7 cm is cut and bend again into an arc of a circle of radius 12 cm. The angle subtended by an arc at the centre of the circle is ____.

Solution:

Given the diameter of circular wire = 14 cm. Therefore, length of wire = 14π cm. Hence,

$$\text{Required angle} = \frac{\text{Arc}}{\text{Radius}} = \frac{14\pi}{12} = \frac{7\pi}{6} \text{ radian}$$

Illustration 2.5 The angles of a quadrilateral are in AP and the greatest angle is 120°. The angles in radians are _____.

Solution:

Let the angles in degrees be $\alpha - 3\delta,\ \alpha - \delta,\ \alpha + \delta,\ \alpha + 3\delta$.

Sum of the angles $= 4\alpha = 360° \Rightarrow \alpha = 90°$

Greatest angle $= \alpha + 3\delta = 120°$

Hence,

$$3\delta = 120° - \alpha = 120° - 90° = 30°$$
$$\Rightarrow \delta = 10°$$

Hence, the angles in degrees are

$$90° - 30° = 60°;\ 90° - 10° = 80°$$
$$90° + 10° = 100°;\ 90° + 30° = 120°$$

In terms of radians, the angles are $\frac{\pi}{3}, \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{2\pi}{3}$.

2.6 Trigonometric Ratio or Function

The six trigonometric ratios sine, cosine, tangent, cotangent, secant and cosecant of an angle θ, $0° < \theta < 90°$, are defined as the ratios of two sides of a right-angled triangle with θ as the angle between base and hypotenuse. However, these can be defined through a unit circle more elegantly.

Draw a unit circle and take any two diameters at right angle as X and Y (Fig. 2.6). Taking OX as the initial line, let $\overline{OP}$ be the radius vector corresponding to an angle θ, where P lies on the unit circle. Let (x, y) be the coordinates of P.

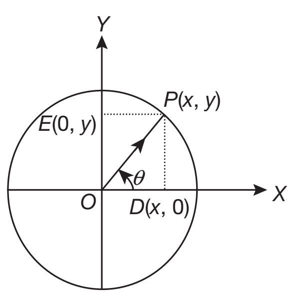

Figure 2.6

Then by definition

$$\cos\theta = x, \text{ the } x\text{-coordinate of } P$$
$$\sin\theta = y, \text{ the } y\text{-coordinate of } P$$

$$\tan\theta = \frac{y}{x}, x \neq 0$$

$$\cot\theta = \frac{x}{y}, y \neq 0$$

$$\sec\theta = \frac{1}{x}, x \neq 0$$

$$\operatorname{cosec}\theta = \frac{1}{y}, y \neq 0$$

Angles measured anticlockwise from the initial line OX are deemed to be positive and angles measured clockwise are considered to be negative.

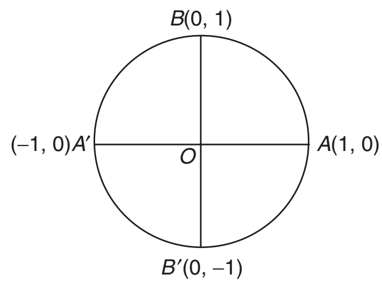

Figure 2.7

Since we can associate a unique radius vector $\overrightarrow{OP}$ and a unique point P with each angle θ, we say x and y and their ratios are functions of θ. This justifies the term 'trigonometric function'. This definition holds good for all angles positive, negative, acute or not acute (irrespective of the magnitude of the angle).

This definition also helps us to write the sine and cosine of four important angles 0°, 90°, 180° and 270° easily (see Fig. 2.7).

$$\theta = 0° \Rightarrow A(1, 0)$$
$$\theta = 90° \Rightarrow B(0, 1)$$
$$\theta = 180° \Rightarrow A'(-1, 0)$$
$$\theta = 270° \Rightarrow B'(0, -1)$$

$$\left.\begin{matrix}\cos 0° = 1\\ \sin 0° = 0\end{matrix}\right\} \quad \left.\begin{matrix}\cos 90° = 0\\ \sin 90° = 1\end{matrix}\right\} \quad \left.\begin{matrix}\cos 180° = -1\\ \sin 180° = 0\end{matrix}\right\} \quad \left.\begin{matrix}\cos 270° = 0\\ \sin 270° = -1\end{matrix}\right\}$$

We can also infer the quadrant rule for sine, cosine and tangent easily.

I quadrant	**II quadrant**	**III quadrant**	**IV quadrant**
sin, cosine and tangent **is positive**	sin alone **is positive**	tangent alone **is positive**	cosine alone **is positive**

$$90° \rightarrow \text{Point } B(0, 1)$$

Since, $\tan\theta = y/x$, $x \neq 0$, $\tan 90° = 1/0$ and hence undefined. However, as θ increases from 0 to 90°, $\tan\theta$ increases from 0 to $+\infty$.

Similarly, sec 90°, cot 0°, cosec 0° are also undefined. 360° and 0° correspond to one and the same point $A(1, 0)$. Therefore, the trigonometric functions of 360° are the same as trigonometric functions of 0°.

$$\sin 360° = 0, \cos 360° = 1 \text{ and } \tan 360° = 0$$

Since $\theta, 2\pi + \theta, 4\pi + \theta, 6\pi + \theta, \ldots, 2n\pi + \theta$ and $\theta - 2\pi, \theta - 4\pi, \theta - 6\pi, \ldots, \theta - 2n\pi$, all correspond to the same radius vector, the trigonometric functions of all these angles are the same as those of θ. Therefore,

$$\sin(2n\pi + \theta) = \sin\theta \text{ and } \sin(\theta - 2n\pi) = \sin\theta$$
$$\cos(2n\pi + \theta) = \cos\theta \text{ and } \cos(\theta - 2n\pi) = \cos\theta$$
$$\tan(2n\pi + \theta) = \tan\theta \text{ and } \tan(\theta - 2n\pi) = \tan\theta$$

The range of the trigonometric ratios in the four quadrants is depicted in the following table.

In the second quadrant (Y)

sine	decreases from	1 to 0
cosine	decreases from	0 to –1
tangent	increases from	$-\infty$ to 0
cotangent	decreases from	0 to $-\infty$
secant	increases from	$-\infty$ to – 1
cosecant	increases from	1 to ∞

In the first quadrant

sine	increases from	0 to 1
cosine	decreases from	1 to 0
tangent	increases from	0 to ∞
cotangent	decreases from	∞ to 0
secant	increases from	1 to ∞
cosecant	decreases from	∞ to 1

X′ *O* *X*

In the third quadrant

sine	decreases from	0 to –1
cosine	increases from	–1 to 0
tangent	increases from	0 to ∞
cotangent	decreases from	∞ to 0
secant	decreases from	–1 to $-\infty$
cosecant	increases from	$-\infty$ to –1

In the fourth quadrant

sine	increases from	–1 to 0
cosine	increases from	0 to 1
tangent	increases from	$-\infty$ to 0
cotangent	decreases from	0 to $-\infty$
secant	decreases from	∞ to 1
cosecant	decreases from	–1 to $-\infty$

Y′

2.6.1 Trigonometric Functions of $-\theta$

Let *OP* and *OP′* be the radii vectors on the unit circle corresponding to θ and $-\theta$. If (x, y) are the coordinates of *P*, then $(x, -y)$ would be the coordinates of *P′*. Now, $\sin\theta = y$ and $\sin(-\theta) = -y$. Hence,

$$\sin(-\theta) = -\sin\theta$$

Similarly,

$$\cos(-\theta) = \cos\theta \text{ and } \tan(-\theta) = -\tan\theta$$

2.6.2 Circular Functions of Allied Angles

When θ is an acute angle, $90° - \theta$ is called the **angle complementary** to θ. Trigonometric functions of $90° - \theta$ are related to trigonometric functions of θ as follows:

$\sin(90° - \theta) = \cos\theta$ $\text{cosec}(90° - \theta) = \sec\theta$
$\cos(90° - \theta) = \sin\theta$ $\sec(90° - \theta) = \text{cosec}\,\theta$
$\tan(90° - \theta) = \cot\theta$ $\cot(90° - \theta) = \tan\theta$

When θ is acute, θ and $180° - \theta$ are called **supplementary angles**.

$\sin(180° - \theta) = \sin\theta$ $\text{cosec}(180° - \theta) = \text{cosec}\,\theta$
$\cos(180° - \theta) = -\cos\theta$ $\sec(180° - \theta) = -\sec\theta$
$\tan(180° - \theta) = -\tan\theta$ $\cot(180° - \theta) = -\cot\theta$

Formulae for the functions of $180° + \theta$, $270° - \theta$, $270° + \theta$, $360° - \theta$ can all be derived with the help of unit circle definition.

There is an easy way to remember these formulae. First of all think of θ as an acute angle. Angles like $180° \pm \theta$, $360° \pm \theta$, $-\theta$ can be considered as angles associated with the horizontal line, angles like $90° - \theta$, $90° + \theta$, $270° \mp \theta$ can be considered as angles associated with vertical line. When associated with the horizontal line, the magnitude of the function does not change, whereas with the vertical line the function changes to the corresponding complementary value. For example, $\sin(180° + \theta)$ will be only $\sin\theta$ (in magnitude) plus or minus and $\cos(180° - \theta)$ will be cosine θ only in magnitude.

To decide upon the sign, consider the quadrant in which the angle falls and decide the sign by the quadrant rule.

For example, $\sin(180° + \theta)$ is $\sin\theta$ (in magnitude), $(180° + \theta)$ lies in third quadrant and hence $\sin(180° + \theta)$ is negative. Therefore

$$\sin(180° + \theta) = -\sin\theta$$

Now consider $\cos(360° - \theta)$: first of all, it should be $\cos\theta$ (in magnitude); since $(360° - \theta)$ lies in IV quadrant, its cosine is positive. Hence,

$$\cos(360° - \theta) = \cos\theta$$

Again consider $\tan(90° + \theta)$: This should be $\cot\theta$ and must have a negative sign since $(90° + \theta)$ is in II quadrant and hence $\tan(90° + \theta)$ is negative. Hence,

$$\tan(90° + \theta) = -\cot\theta$$

Following is the table of formulae for allied angles.

	$180° - \theta$	$180° + \theta$	$360° - \theta$	$-\theta$	$90° - \theta$	$90° + \theta$	$270° - \theta$	$270° + \theta$
sin	$\sin\theta$	$-\sin\theta$	$-\sin\theta$	$-\sin\theta$	$\cos\theta$	$\cos\theta$	$-\cos\theta$	$-\cos\theta$
cos	$-\cos\theta$	$-\cos\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\sin\theta$	$\sin\theta$
tan	$-\tan\theta$	$\tan\theta$	$-\tan\theta$	$-\tan\theta$	$\cot\theta$	$-\cot\theta$	$\cot\theta$	$-\cot\theta$

These formulae are not memorized but derived as and when the occasion demands according to the rule explained above.

Trigonometric ratios of 30°, 45° and 60° are of great importance in solving problems on heights and distances. These along with 0° and 90° are written in tabular form and remembered.

ANGLE / RATIO	0°	30°	45°	60°	90°
sine	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1
cosine	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0
tangent	0	$1/\sqrt{3}$	1	$\sqrt{3}$	undefined
cotangent	undefined	$\sqrt{3}$	1	$1/\sqrt{3}$	0
secant	1	$2/\sqrt{3}$	$\sqrt{2}$	2	undefined
cosecant	undefined	2	$\sqrt{2}$	$2/\sqrt{3}$	1

2.6.3 Important Facts of Trigonometric Functions

The following points may be noted:

1. For any power n, $(\sin A)^n$ is written as $\sin^n A$. Similarly, for all other trigonometric ratios.
2. cosec A, sec A and cot A are, respectively, the reciprocals of sin A, cos A and tan A.
3. (a) $\sin^2 A + \cos^2 A = 1$
   (b) $1 + \tan^2 A = \sec^2 A$
   (c) $1 + \cot^2 A = \text{cosec}^2 A$.
4. sec A − tan A and sec A + tan A are reciprocals. So also are cosec A − cot A and cosec A + cot A.
   Whenever sec A or tan A is thought of for an angle A, it is necessary to stress that, $A \neq \pi/2$ particularly, and generally $A \neq n\pi + \pi/2$, $n \in \boldsymbol{N}$, where $\boldsymbol{N}$ is the set of natural numbers.
5. sin A and cos A are bounded functions which can be seen from the following inequalities:
   (a) $|\sin A| \leq 1 \Rightarrow -1 \leq \sin A \leq 1$
   (b) $|\cos A| \leq 1 \Rightarrow -1 \leq \cos A \leq 1$
   (c) $|\text{cosec}\, A| \geq 1 \Rightarrow \text{cosec}\, A \geq 1$ or $\text{cosec}\, A \leq -1$
   (d) $|\sec A| \geq 1 \Rightarrow \sec A \geq 1$ or $\sec A \leq -1$

   (a) $\sin\left(\frac{\pi}{2} - A\right) = \sin\left(\frac{\pi}{2} + A\right) = \cos A$
   (b) $\cos\left(\frac{\pi}{2} - A\right) = -\cos\left(\frac{\pi}{2} + A\right) = \sin A$
   (c) $\sin(\pi - A) = -\sin(\pi + A) = \sin A$
   (d) $\cos(\pi - A) = \cos(\pi + A) = -\cos A$
   (e) $\tan(\pi - A) = -\tan(\pi + A) = -\tan A$
6. The trigonometric ratios are also called trigonometric functions. They are also sometimes called **circular functions**.

   The trigonometric functions, apart from possessing many other properties, exhibit a property of the values being repeated when the angle is changed (increased or decreased) by a constant value. Such a property is referred to as **periodicity**. Thus,

$$\begin{aligned}\sin x &= \sin(x + 2\pi) = \sin(x + 4\pi)\\ &= \sin(x - 2\pi) = \sin(x + 2k\pi),\ k \text{ an integer}\\ \cos x &= \cos(x + 2\pi) = \cos(x + 4\pi)\\ &= \cos(x - 2\pi) = \cos(x + 2k\pi),\ k \text{ an integer}\end{aligned}$$

   Hence, both $\sin x$ and $\cos x$ are periodic functions of period 2π radians. From point 5, it is clear that they are also bounded functions.
   Note that:
   1. cosec x and sec x, whenever they exist, are also periodic of period 2π radians.
   2. tan x and cot x, when they exist, are periodic of period π radians.
   3. tan x, sec x, cosec x and cot x are unbounded functions.

2.6.4 Graphs of Trigonometric Functions

1. $y = \sin x$ (Fig. 2.8)

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
sin x	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	−1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0

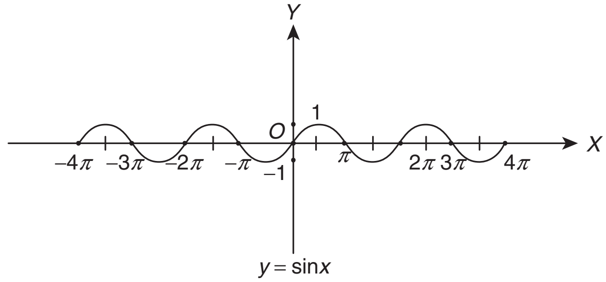

Figure 2.8

2. $y = \cos x$ (Fig. 2.9)

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
cos x	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{3}}{2}$	−1	$\frac{-\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1

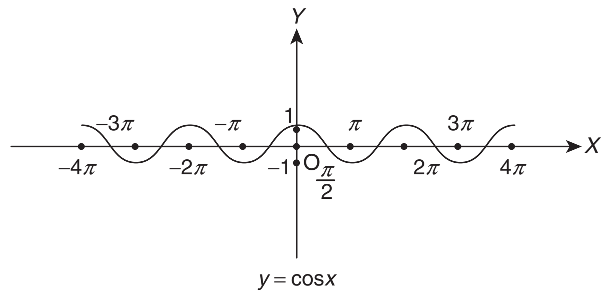

Figure 2.9

3. $y = \tan x$ (Fig. 2.10)

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
tan x	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$-\sqrt{3}$	−1	$-\frac{1}{\sqrt{3}}$	0

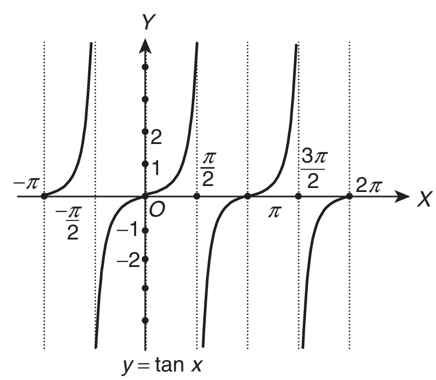

Figure 2.10

4. $y = \cot x$ (Fig. 2.11)

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
cot x	undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	−1	$-\sqrt{3}$	undefined

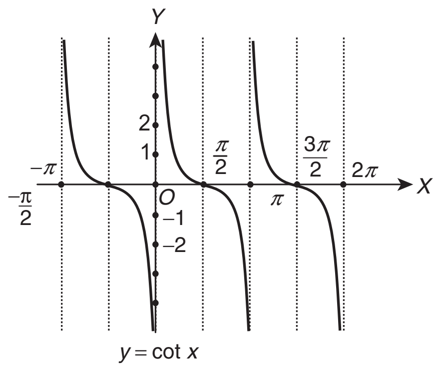

Figure 2.11

5. $y = \sec x$ (Fig. 2.12)

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
sec x	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	−2	$-\sqrt{2}$	$\frac{-2}{\sqrt{3}}$	−1

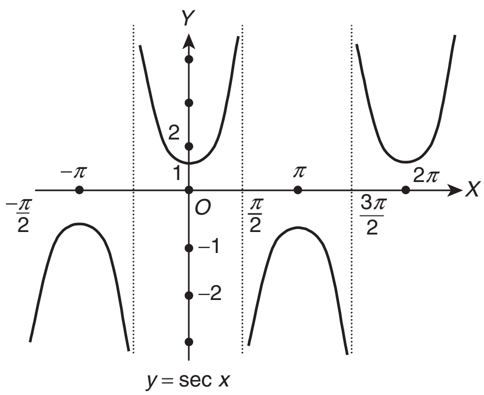

Figure 2.12

6. $y = \text{cosec}\, x$ (Fig. 2.13)

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
cosec x	undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined

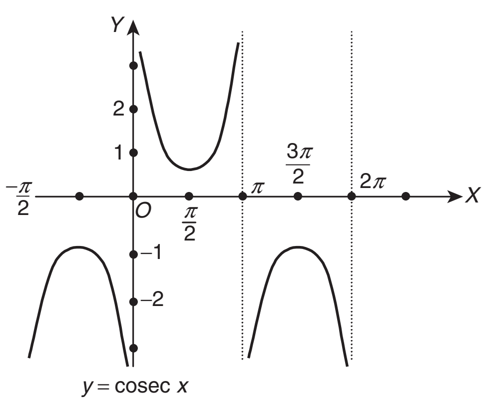

Figure 2.13

Illustration 2.6 Evaluate:
1. sin (1560°)
2. cos (−3030°)

Solution:

1. $\sin(1560°) = \sin(4 \times 360° + 120°) = \sin 120° = \sin(180° - 60°)$
$= \sin 60° = \frac{\sqrt{3}}{2}$

2. $\cos(-3030°) = \cos(3030°)$ [using $\cos(-\theta) = \cos\theta$]
$= \cos(8 \times 360° + 150°) = \cos 150° = \cos(180° - 30°)$
$= -\cos 30° = -\frac{\sqrt{3}}{2}$

Illustration 2.7 Prove that $(1 - \sin\theta + \cos\theta)^2 = 2(1 - \sin\theta)(1 + \cos\theta)$.

Solution:

L.H.S. $= [(1 - \sin\theta) + \cos\theta]^2 = (1 - \sin\theta)^2 + \cos^2\theta + 2\cos\theta(1 - \sin\theta)$
$= (1 - \sin\theta)^2 + (1 - \sin^2\theta) + 2\cos\theta(1 - \sin\theta)$
$= (1 - \sin\theta)\cdot[(1 - \sin\theta) + (1 + \sin\theta) + 2\cos\theta]$
$= (1 - \sin\theta)\cdot(2 + 2\cos\theta) = 2(1 - \sin\theta)(1 + \cos\theta)$

Illustration 2.8 Prove that $\text{cosec}^4\theta(1 - \cos^4\theta) = 1 + 2\cot^2\theta$.

Solution:

$\text{cosec}^4\theta\,(1 - \cos^4\theta) - 2\cot^2\theta = \frac{\text{cosec}^2\theta(1-\cos^2\theta)(1+\cos^2\theta)}{\sin^2\theta}$
$- 2\cot^2\theta$

$= \text{cosec}^2\theta(1 + \cos^2\theta) - 2\cot^2\theta = \text{cosec}^2\theta + \cot^2\theta - 2\cot^2\theta$
$= 1 + 2\cot^2\theta - 2\cot^2\theta = 1$

Illustration 2.9 Find the minimum and maximum values of $\sin^2\theta + \cos^4\theta$.

Solution: The given expression can be written as

$$\sin^2\theta + \cos^4\theta = 1 - \cos^2\theta + \cos^4\theta$$

It can be considered as a quadratic in $\cos^2\theta$. So we have

$$\sin^2\theta + \cos^4\theta = 1 - \cos^2\theta + \cos^4\theta.$$

$$= 1 + \left(\cos^2\theta - \frac{1}{2}\right)^2 - \frac{1}{4}$$

$$= \frac{3}{4} + \left(\cos^2\theta - \frac{1}{2}\right)^2 \geq \frac{3}{4}$$

Hence, the expression has a minimum value 3/4.
Also

$$\sin^2\theta + \cos^4\theta = \sin^2\theta + \cos^2\theta\cos^2\theta \leq \sin^2\theta + \cos^2\theta = 1$$

Therefore, maximum value = 1.

Illustration 2.10 For any real θ, find the maximum value of $\cos^2(\cos\theta) + \sin^2(\sin\theta)$.

Solution:

$$-1 \leq \cos\theta \leq 1 \Rightarrow \cos 1 \leq \cos(\cos\theta) \leq 1$$
$$\Rightarrow \cos^2 1 \leq \cos^2(\cos\theta) \leq 1 \quad (1)$$
$$-1 \leq \sin\theta \leq 1 \Rightarrow -\sin 1 \leq \sin(\sin\theta) \leq \sin 1$$
$$\Rightarrow 0 \leq \sin^2(\sin\theta) \leq \sin^2 1 \quad (2)$$

From Eqs. (1) and (2) we can see that maximum value of $\cos^2(\cos\theta) + \sin^2(\sin\theta)$ exists at $\theta = \pi/2$ which is $1 + \sin^2 1$.
Hence, maximum value is $1 + \sin^2 1$.

Illustration 2.11 If $2\tan^2\alpha\tan^2\beta\tan^2\gamma + \tan^2\alpha\tan^2\beta + \tan^2\beta\tan^2\gamma + \tan^2\gamma\tan^2\alpha = 1$, prove that $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 1$.

Solution: We have

$2\tan^2\alpha\tan^2\beta\tan^2\gamma + \tan^2\alpha\tan^2\beta + \tan^2\beta\tan^2\gamma + \tan^2\gamma\tan^2\alpha = 1.$

So dividing both sides by $\tan^2\alpha\tan^2\beta\tan^2\gamma$, we get

$$2 + \cot^2\gamma + \cot^2\alpha + \cot^2\beta = \cot^2\alpha\cot^2\beta\cot^2\gamma$$

$$\Rightarrow \text{cosec}^2\alpha + \text{cosec}^2\beta + \text{cosec}^2\gamma - 1 = (\text{cosec}^2\alpha - 1)(\text{cosec}^2\beta - 1)(\text{cosec}^2\gamma - 1)$$

$$\Rightarrow \text{cosec}^2\alpha + \text{cosec}^2\beta + \text{cosec}^2\gamma - 1 = -1 + \text{cosec}^2\alpha + \text{cosec}^2\beta + \text{cosec}^2\gamma - (\text{cosec}^2\alpha\,\text{cosec}^2\beta + \text{cosec}^2\beta\,\text{cosec}^2\gamma + \text{cosec}^2\gamma\,\text{cosec}^2\alpha + \text{cosec}^2\alpha\cdot\text{cosec}^2\beta\cdot\text{cosec}^2\gamma\,\text{cosec}^2\alpha\,\text{cosec}^2\beta + \text{cosec}^2\beta\cdot\text{cosec}^2\gamma + \text{cosec}^2\gamma\,\text{cosec}^2\alpha$$

$$= \text{cosec}^2\alpha\,\text{cosec}^2\beta\cdot\text{cosec}^2\gamma$$

$$\Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 1$$

Illustration 2.12 Simplify $\dfrac{\sin\left(\dfrac{3\pi}{2}-\theta\right)\cos\left(\dfrac{\pi}{2}+\theta\right)}{\tan\left(\dfrac{\pi}{2}+\theta\right)} - \dfrac{\sin\left(\dfrac{3\pi}{2}-\theta\right)}{\sec(\pi+\theta)}$.

Solution:

The expression can be rewritten as

$$\frac{(-\cos\theta)(-\sin\theta)}{(-\cot\theta)} - \frac{(-\cos\theta)}{(-\sec\theta)} = -\sin^2\theta - \cos^2\theta = -1$$

2.6.5 Circular Function of Compound Angle

An equation involving trigonometric functions, which is true for all those values of θ for which the functions are defined, is called a trigonometric identity; otherwise it is a trigonometric equation.

We shall now derive some results which are useful in simplifying trigonometric equations.

To prove:

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi \qquad (2.4)$$

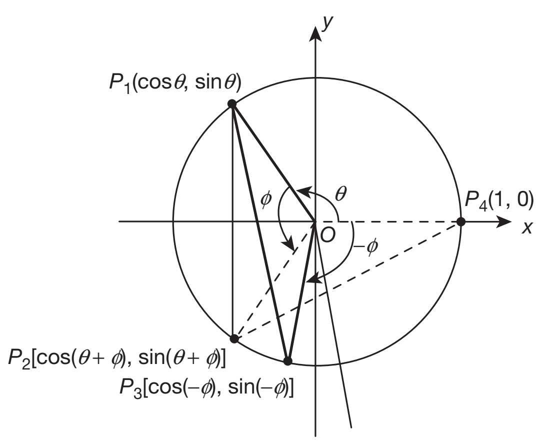

Figure 2.14

Consider a unit circle with origin as its centre (Fig. 2.14). Let

$$\angle P_4OP_1 = \theta \text{ and } \angle P_1OP_2 = \phi$$

Hence,

$$\angle P_4OP_2 = \theta + \phi \text{ and } \angle P_4OP_3 = -\phi$$

Coordinates of P_1, P_2, P_3, P_4 are

$$P_1(\cos\theta, \sin\theta)$$
$$P_2[\cos(\theta+\phi), \sin(\theta+\phi)]$$
$$P_3[\cos(-\phi), \sin(-\phi)]$$
$$P_4(1, 0)$$

ΔP_1OP_3 is congruent to ΔP_2OP_4.

Since $OP_1 = OP_4 = OP_3 = OP_2$ = Radius of the circle

$$\angle P_1OP_3 = \angle P_2OP_4 = 360° - (\theta + \phi)$$

Therefore, by side angle, the triangles are congruent. Hence,

$$P_1P_3 = P_2P_4$$

Applying the distance formula,

$$P_1P_3^{\,2} = [\cos\theta - \cos(-\phi)]^2 + [\sin\theta - \sin(-\phi)]^2$$
$$= (\cos\theta - \cos\phi)^2 + (\sin\theta + \sin\phi)^2$$
$$[\text{using } \cos(-\phi) = \cos\phi \text{ and } \sin(-\phi) = -\sin\phi]$$
$$= \cos^2\theta + \cos^2\phi - 2\cos\theta\cos\phi + \sin^2\theta + \sin^2\phi + 2\sin\theta\sin\phi$$
$$= 2 - 2(\cos\theta\cos\phi - \sin\theta\sin\phi)$$
$$P_2P_4^2 = [1 - \cos(\theta+\phi)]^2 + [0 - \sin(\theta+\phi)]^2$$
$$= 1 - 2\cos(\theta+\phi) + \cos^2(\theta+\phi) + \sin^2(\theta+\phi)$$
$$= 2 - 2\cos(\theta+\phi)$$

Since $P_1P_3 = P_2P_4$, we have $P_1P_3^2 = P_2P_4^2$. Therefore

$$2 - 2(\cos\theta\cos\phi - \sin\theta\sin\phi) = 2 - 2\cos(\theta+\phi)$$

Hence,

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

Replacing ϕ by $-\phi$ in Eq. (2.4), we get

$$\cos(\theta - \phi) = \cos\theta\cos(-\phi) - \sin\theta\sin(-\phi)$$

or

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi \qquad (2.5)$$

2.7 Formulae for Trigonometric Ratios of Sum and Differences of Two or More Angles

1. $\sin(A+B) = \sin A\cos B + \cos A\sin B$
2. $\sin(A-B) = \sin A\cos B - \cos A\sin B$
3. $\cos(A+B) = \cos A\cos B - \sin A\sin B$
4. $\cos(A-B) = \cos A\cos B + \sin A\sin B$
5. $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$
6. $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$
7. $\cot(A+B) = \dfrac{\cot A\cot B - 1}{\cot A + \cot B}$
8. $\cot(A-B) = \dfrac{\cot A\cot B + 1}{\cot B - \cot A}$
9. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

10. $\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B=\cos^2 B-\sin^2 A$

11. $\sin(A+B+C)=\sin A\cos B\cos C+\sin B\cos A\cos C$
$$+\sin C\cos A\cos C-\sin A\sin B\sin C$$
$$=\cos A\cos B\cos C(\tan A+\tan B+\tan C-\tan A\tan B\tan C)$$

12. $\cos(A+B+C)=\cos A\cos B\cos C-\sin A\sin B\cos C$
$$-\sin A\cos B\sin C-\cos A\sin B\sin C$$
$$=\cos A\cos B\cos C(1-\tan A\tan B-\tan B\tan C-\tan C\tan A)$$

13. $\tan(A+B+C)=\dfrac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan C\tan B-\tan A\tan C}$

14. $\cot(A+B+C)=\dfrac{\cot A\cot B\cot C-\cot A-\cot B-\cot C}{\cot A\cot B+\cot B\cot C+\cot C\cot A-1}$

Illustration 2.13 $\dfrac{\cos 12^\circ-\sin 12^\circ}{\cos 12^\circ+\sin 12^\circ}+\dfrac{\sin 147^\circ}{\cos 147^\circ}=$ ____.

Solution:

$$\frac{\cos 12^\circ-\sin 12^\circ}{\cos 12^\circ+\sin 12^\circ}+\frac{\sin 147^\circ}{\cos 147^\circ}=\frac{1-\tan 12^\circ}{1+\tan 12^\circ}+\tan 147^\circ$$
$$=\tan(45^\circ-12^\circ)+\tan(180^\circ-33^\circ)$$
$$=\tan 33^\circ+(-\tan 33^\circ)=0$$

Illustration 2.14 Solve $\dfrac{\sin^2 A-\sin^2 B}{\sin A\cos A-\sin B\cos B}$.

Solution:

$$\frac{2(\sin^2 A-\sin^2 B)}{2\sin A\cos A-2\sin B\cos B}=\frac{2\sin(A+B)\cdot\sin(A-B)}{\sin 2A-\sin 2B}$$
$$=\frac{2\sin(A+B)\sin(A-B)}{2\sin(A-B)\cos(A+B)}=\tan(A+B)$$

Illustration 2.15 If $\tan\theta-\cot\theta=a$ and $\sin\theta+\cos\theta=b$, then solve $(b^2-1)^2(a^2+4)$.

Solution:

Given that

$$\tan\theta-\cot\theta=a \quad (1)$$

and

$$\sin\theta+\cos\theta=b \quad (2)$$

Now,

$$(b^2-1)^2(a^2+4)=[(\sin\theta+\cos\theta)^2-1]^2[(\tan\theta-\cot\theta)^2+4]$$
$$=[1+\sin 2\theta-1]^2[\tan^2\theta+\cot^2\theta-2+4]=\sin^2 2\theta(\operatorname{cosec}^2\theta+\sec^2\theta)$$
$$=4\sin^2\theta\cos^2\theta\left[\frac{1}{\sin^2\theta}+\frac{1}{\cos^2\theta}\right]=4$$

Trick: Obviously the value of expression $(b^2-1)^2(a^2+4)$ is independent of θ, therefore put any suitable value of θ. Let $\theta=45^\circ$. We get $a=0$, $b=\sqrt{2}$ so that $[(\sqrt{2})^2-1]^2(0^2+4)=4$.

Illustration 2.16 If $\sin\theta=\dfrac{8}{17}$ and $\cos\beta=\dfrac{9}{41}$, find $\sin(\theta+\beta)$, $\cos(\theta+\beta)$, $\sin(\theta-\beta)$ and $\cos(\theta-\beta)$, where θ is an obtuse angle and β is an acute angle.

Solution:

Since $\sin\theta=\dfrac{8}{17}$, we have

$$\cos^2\theta=1-\sin^2\theta=1-\left(\frac{8}{17}\right)^2=1-\frac{64}{289}=\frac{225}{289}$$

Therefore,

$$\cos\theta=\pm\frac{15}{17}$$

As θ is obtuse, $\cos\theta$ is negative. Therefore, $\cos\theta=-\dfrac{15}{17}$.

Now $\cos\beta=9/41$ and $\sin^2\beta=\cos^2\beta-1$. So

$$\sin^2\beta=1-\frac{81}{1681}=\frac{1600}{1681}$$
$$\Rightarrow \sin\beta=\pm\frac{40}{41}$$

As β is acute, $\sin\beta$ is positive. Hence

$$\sin\beta=+\frac{40}{41}$$

Now

$$\sin(\theta+\beta)=\sin\theta\cos\beta+\cos\theta\sin\beta$$
$$=\frac{8}{17}\cdot\frac{9}{41}+\left(-\frac{15}{17}\right)\cdot\frac{40}{41}=-\frac{528}{697}$$
$$\cos(\theta+\beta)=\cos\theta\cos\beta-\sin\theta\sin\beta$$
$$=\left(-\frac{15}{17}\right)\cdot\frac{9}{41}-\left(\frac{8}{17}\right)\cdot\frac{40}{41}=-\frac{455}{697}$$
$$\sin(\theta-\beta)=\sin\theta\cos\beta-\cos\theta\sin\beta$$
$$=\left(\frac{8}{17}\right)\cdot\frac{9}{41}-\left(-\frac{15}{17}\right)\cdot\frac{40}{41}=\frac{672}{697}$$
$$\cos(\theta-\beta)=\cos\theta\cos\beta+\sin\theta\sin\beta$$
$$=\left(-\frac{15}{17}\right)\cdot\frac{9}{41}+\left(\frac{8}{17}\right)\cdot\frac{40}{41}=\frac{185}{697}$$

Illustration 2.17 Consider triangle ABC in which $A+B+C=\pi$. Prove that

1. $\tan A+\tan B+\tan C=\tan A\tan B\tan C$
2. $\tan(B/2)\tan(C/2)+\tan(C/2)\tan(A/2)+\tan(A/2)\tan(B/2)=1$

Solution:

1. We have $A+B=\pi-C=180^\circ-C$
$$\Rightarrow \tan(A+B)=\tan(180^\circ-C)=-\tan C$$
$$\Rightarrow \frac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C$$
$$\Rightarrow \tan A+\tan B=-\tan C(1-\tan A\tan B)$$
$$\Rightarrow \tan A+\tan B+\tan C=\tan A\tan B\tan C$$

2. We have $(A/2+B/2)=\pi/2-C/2=90^\circ-C/2$
$$\Rightarrow \tan(A/2+B/2)=\tan(\pi/2-C/2)=\cot(C/2)$$

$$\Rightarrow \frac{\tan A/2 + \tan B/2}{1 - \tan A/2 \tan B/2} = \cot(C/2)$$

$$\Rightarrow \tan\frac{C}{2}\,(\tan A/2 + \tan B/2) = 1 - \tan\frac{A}{2}\tan\frac{B}{2}$$

Therefore, we get

$$\tan(C/2)\tan(A/2) + \tan(B/2)\tan(C/2) + \tan(A/2)\tan(B/2) = 1$$

Your Turn 1

1. If $\frac{\sin^3\theta}{\sin(2\theta+\alpha)} = \frac{\cos^3\theta}{\cos(2\theta+\alpha)}$, prove that $\tan 2\theta = 2\tan(3\theta+\alpha)$.
2. If $\tan\alpha = \frac{m}{m+1}$ and $\tan\beta = \frac{1}{2m+1}$ then find $\alpha+\beta$.

   Ans. $\alpha+\beta = \frac{\pi}{4}$
3. Prove that $\tan(112A)\tan(99A)\tan(13A) = \tan(112A) - \tan(99A) - \tan(13A)$.
4. If $\frac{x}{\cos\theta} = \frac{y}{\cos\left(\theta+\frac{2\pi}{3}\right)} = \frac{z}{\cos\left(\theta+\frac{4\pi}{3}\right)}$, then $x+y+z$ is equal to ____. **Ans.** 0
5. If $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 3$, then $\cos\theta_1 + \cos\theta_2 + \cos\theta_3 =$ ____. **Ans.** 0

2.8 Formulae to Transform Product into Sum or Difference

1. $2\sin A\cos B = \sin(A+B) + \sin(A-B)$
2. $2\cos A\sin B = \sin(A+B) - \sin(A-B)$
3. $2\cos A\cos B = \cos(A+B) + \cos(A-B)$
4. $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

Illustration 2.18 Show that 8 sin 10° sin 50° sin 70° = 1.

Solution:

$$\begin{aligned}\text{L.H.S.} &= 4\,(2\sin 50^\circ \sin 10^\circ)\sin 70^\circ\\ &= 4[\cos(50^\circ - 10^\circ) - \cos(50^\circ + 10^\circ)]\sin 70^\circ\\ &= 2\cdot(2\sin 70^\circ\cdot\cos 40^\circ) - 4\cos 60^\circ \sin 70^\circ\\ &= 2\sin 70^\circ + 2\sin 30^\circ - 2\sin 70^\circ\\ &= 2\sin 30^\circ = 1\end{aligned}$$

Illustration 2.19 Show that $\sin(45^\circ + A)\sin(45^\circ - A) = \frac{1}{2}\cos 2A$

Solution:

$$\sin(45^\circ + A)\sin(45^\circ - A) = \frac{1}{2}\{[\cos(2A) - \cos(90^\circ)]\} = \frac{1}{2}\cos(2A)$$

2.9 Formulae to Transform Sum or Difference into Product

Let $A+B = C$ and $A-B = D$. Then

$$A = \frac{C+D}{2} \text{ and } B = \frac{C-D}{2}$$

Therefore,

1. $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$
2. $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$
3. $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$
4. $\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

Illustration 2.20 If $\sin B = \frac{1}{5}\sin(2A+B)$, then $\frac{\tan(A+B)}{\tan A} =$ ____.

Solution:

$$\frac{\sin(2A+B)}{\sin B} = \frac{5}{1}$$

By Componendo and Dividendo, we have

$$\frac{\sin(2A+B) + \sin B}{\sin(2A+B) - \sin B} = \frac{5+1}{5-1}$$

$$\frac{2\sin(A+B)\cdot\cos A}{2\cos(A+B)\cdot\sin A} = \frac{6}{4} \Rightarrow \frac{\tan(A+B)}{\tan A} = \frac{3}{2}$$

Illustration 2.21 Solve $\frac{\sin 70^\circ + \cos 40^\circ}{\cos 70^\circ + \sin 40^\circ}$.

Solution:

$$\frac{\sin 70^\circ + \cos 40^\circ}{\cos 70^\circ + \sin 40^\circ} = \frac{\sin 70^\circ + \sin 50^\circ}{\sin 20^\circ + \sin 40^\circ} = \frac{2\sin 60^\circ \cos 10^\circ}{2\sin 30^\circ \cos(-10^\circ)}$$

$$= \frac{\sin 60^\circ}{\sin 30^\circ} = \frac{\sqrt{3}}{2}\cdot\frac{2}{1} = \sqrt{3}$$

Illustration 2.22 Show that

$$\frac{\sin 7x - \sin 3x - \sin 5x + \sin x}{\cos 7x + \cos 3x - \cos 5x - \cos x} = \tan 2x$$

Solution:

$$\begin{aligned}\text{Numerator} &= (\sin 7x + \sin x) - (\sin 5x + \sin 3x)\\ &= 2\sin 4x\cdot\cos 3x - 2\sin 4x\cdot\cos x \text{ (using C.D. formula)}\\ &= 2\sin 4x\,(\cos 3x - \cos x)\end{aligned}$$

$$\begin{aligned}\text{Denominator} &= (\cos 3x - \cos 5x) - (\cos x - \cos 7x)\\ &= 2\sin 4x\sin x - 2\sin 4x\sin 3x\\ &= 2\sin 4x\,(\sin x - \sin 3x)\end{aligned}$$

Therefore, the given expression is

$$\frac{\cos 3x - \cos x}{\sin x - \sin 3x} = \frac{2\sin 2x\sin x}{2\cos 2x\sin x} = \tan 2x$$

Illustration 2.23 Solve $\sin 47^\circ + \sin 61^\circ - \sin 11^\circ - \sin 25^\circ$.

Solution:

$$\sin 47^\circ + \sin 61^\circ - (\sin 11^\circ + \sin 25^\circ) = 2\sin 54^\circ\cdot\cos 7^\circ - 2\sin 18^\circ\cos 7^\circ$$

$$= 2\cos 7^\circ(\sin 54^\circ - \sin 18^\circ) = 2\cos 7^\circ\cdot 2\cos 36^\circ\cdot\sin 18^\circ$$

$$= 4\cdot\cos 7^\circ\cdot\frac{\sqrt{5}+1}{4}\cdot\frac{\sqrt{5}-1}{4} = \cos 7^\circ$$

Illustration 2.24 If $\cos(A+B)\sin(C+D)=\cos(A-B)\sin(C-D)$, prove that $\cot A\cot B\cot C=\cot D$.

Solution:

We have

$$\cos(A+B)\sin(C+D)=\cos(A-B)\sin(C-D)$$
$$\Rightarrow \frac{\cos(A-B)}{\cos(A+B)}=\frac{\sin(C+D)}{\sin(C-D)}$$
$$\Rightarrow \frac{\cos(A-B)+\cos(A+B)}{\cos(A-B)-\cos(A+B)}=\frac{\sin(C+D)+\sin(C-D)}{\sin(C+D)-\sin(C-D)}$$
$$\Rightarrow \frac{2\cos A\cos B}{2\sin A\sin B}=\frac{2\sin C\cos D}{2\cos C\sin D}$$
$$\Rightarrow \cot A\cot B=\tan C\cot D$$
$$\Rightarrow \cot A\cot B\cot C=\cot D$$

Illustration 2.25 If A, B, C and D are angles of a quadrilateral and $\sin\frac{A}{2}\sin\frac{B}{2}\cdot\sin\frac{C}{2}\cdot\sin\frac{D}{2}=\frac{1}{4}$, prove that $A=B=C=D=\pi/2$.

Solution:

Given that

$$4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2}=1$$
$$\Rightarrow\left[\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right]\left[\cos\left(\frac{C-D}{2}\right)-\cos\left(\frac{C+D}{2}\right)\right]=1$$

Since, $A+B=2\pi-(C+D)$, the above equation becomes

$$\left[\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right]\left[\cos\left(\frac{C-D}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right]=1$$
$$\Rightarrow\cos^2\left(\frac{A+B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\left[\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{C-D}{2}\right)\right]$$
$$+1-\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{C-D}{2}\right)=0$$

This is quadratic equation in cosine which has real roots. So

$$\left[\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{C-D}{2}\right)\right]^2-4\left[1-\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{C-D}{2}\right)\right]\geq 0$$
$$\Rightarrow\left[\cos\left(\frac{A-B}{2}\right)+\cos\left(\frac{C-D}{2}\right)\right]^2\geq 4$$
$$\Rightarrow\left[\cos\left(\frac{A-B}{2}\right)+\cos\left(\frac{C-D}{2}\right)\right]\geq 2$$

Now both $\cos\left(\frac{A-B}{2}\right)$ and $\cos\left(\frac{C-D}{2}\right)\leq 1$. So

$$\cos\left(\frac{A-B}{2}\right)=1=\cos\left(\frac{C-D}{2}\right)\Rightarrow\frac{A-B}{2}=0=\frac{C-D}{2}$$
$$\Rightarrow A=B, C=D$$

Similarly, $A=C, B=D\Rightarrow A=B=C=D=\pi/2$.

Your Turn 2

1. Solve $\dfrac{\cos 10°-\sin 10°}{\cos 10°+\sin 10°}$ **Ans.** cot 55°
2. If $\tan(A+B)=p$ and $\tan(A-B)=q$ then the value of $\tan 2A=$ _____ .
   Ans. $\tan 2A=\dfrac{p+q}{1-pq}$
3. Solve $\sin 163°\cos 347°+\sin 73°\sin 167°$. **Ans.** 1/2
4. The value of $\cot 70°+4\cos 70°$ is _____. **Ans.** $\sqrt{3}$
5. If $\tan\alpha=(1+2^{-x})^{-1}$, $\tan\beta=(1+2^{x+1})^{-1}$, then $\alpha+\beta$ equals _____.
   Ans. $\alpha+\beta=\dfrac{\pi}{4}$
6. The value of $\dfrac{\tan 70°-\tan 20°}{\tan 50°}$. **Ans.** 2

2.10 Trigonometric Ratio of Multiple of Angles

1. $\sin 2A=2\sin A\cos A=\dfrac{2\tan A}{1+\tan^2 A}$
2. $\cos 2A=1-2\sin^2 A=2\cos^2 A-1=\cos^2 A-\sin^2 A=\dfrac{1-\tan^2 A}{1+\tan^2 A}$
3. $\tan 2A=\dfrac{2\tan A}{1-\tan^2 A}$
4. $\sin 3A=3\sin A-4\sin^3 A=4\sin(60°-A)\sin A\sin(60°+A)$
5. $\cos 3A=4\cos^3 A-3\cos A=4\cos(60°-A)\cos A\cos(60°+A)$
6. $\tan 3A=\dfrac{3\tan A-\tan^3 A}{1-3\tan^2 A}=\tan(60°-A)\tan A\tan(60°+A)$
7. $\sin 4A=4\sin A\cos^3 A-4\cos A\sin^3 A$
8. $\cos 4A=8\cos^4 A-8\cos^2 A+1$
9. $\tan 4A=\dfrac{4\tan A-4\tan^3 A}{1-6\tan^2 A+\tan^4 A}$
10. $\sin(A_1+A_2+A_3+\cdots+A_n)=\cos A_1\cos A_2\cdots\cos A_n(S_1-S_3+S_5-S_7\cdots)$
11. $\cos(A_1+A_2+A_3+\cdots+A_n)=\cos A_1\cos A_2\cdots\cos A_n(1-S_2+S_4-S_6\cdots)$
12. $\tan(A_1+A_2+A_3+\cdots+A_n)=\dfrac{S_1-S_3+S_5-S_7+\cdots}{1-S_2+S_4-S_6+\cdots}$

    where

    $S_1=\tan A_1+\tan A_2+\cdots+\tan A_n$
    = The sum of the tangents of the separate angles
    $S_2=\tan A_1\cdot\tan A_2+\tan A_2\cdot\tan A_3\cdots$
    = The sum of the tangents taken two at a time
    $S_3=\tan A_1\cdot\tan A_2\cdot\tan A_3+\tan A_2\cdot\tan A_3\tan A_4+\cdots$
    = Sum of tangents three at a time, and so on

If $A_1=A_2=A_3=\cdots=A_n$ then $S_1=n\tan A, S_2={}^nC_2\tan^2 A$, $S_3={}^nC_3\tan^3 A\ldots$

Illustration 2.26 If $\frac{\tan 3\theta}{\tan\theta} = 4$, then find the value of $\frac{\sin 3\theta}{\sin\theta}$.

Solution:

$$\frac{\tan 3\theta}{\tan\theta} = \frac{(3\tan\theta - \tan^3\theta)}{(1-3\tan^2\theta)\tan\theta} = 4$$

$$\Rightarrow \frac{3-\tan^2\theta}{1-3\tan^2\theta} = 4 \Rightarrow \tan^2\theta = \frac{1}{11}$$

Now,

$$\frac{\sin 3\theta}{\sin\theta} = 3 - 4\sin^2\theta = 3 - 4\left(\frac{1}{1+\cot^2\theta}\right) = \frac{8}{3}$$

Illustration 2.27 Prove that $\frac{\tan\left(\frac{\pi}{4}+A\right)}{\tan\left(\frac{\pi}{4}-A\right)} = \frac{2\cos A + \sin A + \sin 3A}{2\cos A - \sin A - \sin 3A}$.

Solution:

$$\text{L.H.S.} = \frac{\tan\left(\frac{\pi}{4}+A\right)}{\tan\left(\frac{\pi}{4}-A\right)} = \frac{\frac{1+\tan A}{1-\tan A}}{\frac{1-\tan A}{1+\tan A}} = \left(\frac{1+\tan A}{1-\tan A}\right)^2 = \left(\frac{\cos A+\sin A}{\cos A-\sin A}\right)^2$$

$$= \left(\frac{\cos^2 A+\sin^2 A+2\sin A\cos A}{\cos^2 A+\sin^2 A-2\sin A\cos A}\right) = \left(\frac{1+\sin 2A}{1-\sin 2A}\right)$$

$$\text{R.H.S.} = \frac{2\cos A+2\sin 2A\cos A}{2\cos A-2\sin 2A\cos A} = \left(\frac{1+\sin 2A}{1-\sin 2A}\right)$$

Hence, both sides reduce to the same result.

Illustration 2.28 $\frac{\sec 8A-1}{\sec 4A-1}$ equals to ______.

Solution:

$$\frac{1-\cos 8A}{\cos 8A}\cdot\frac{\cos 4A}{1-\cos 4A} = \frac{2\sin^2 4A}{\cos 8A}\cdot\frac{\cos 4A}{2\sin^2 2A}$$

$$= \frac{2\sin 4A\cdot\cos 4A\cdot\sin 4A}{\cos 8A\cdot 2\sin^2 2A} = \frac{\sin 8A\cdot 2\sin 2A\cdot\cos 2A}{\cos 8A\cdot 2\sin^2 2A} = \frac{\tan 8A}{\tan 2A}$$

2.11 Trigonometric Ratio of Sub-Multiple of Angles

1. $\left|\cos\frac{A}{2}+\sin\frac{A}{2}\right| = \sqrt{1+\sin A}$ or $\cos\frac{A}{2}+\sin\frac{A}{2} = \pm\sqrt{1+\sin A}$

   That is, $\begin{cases} +, \text{ if } 2n\pi - \frac{\pi}{4} \le \frac{A}{2} \le 2n\pi + \frac{3\pi}{4} \\ -, \text{ otherwise} \end{cases}$

2. $\left|\sin\frac{A}{2}-\cos\frac{A}{2}\right| = \sqrt{1-\sin A}$ or $\sin\frac{A}{2}-\cos\frac{A}{2} = \pm\sqrt{1-\sin A}$

   That is, $\begin{cases} +, \text{ if } 2n\pi + \frac{\pi}{4} \le \frac{A}{2} \le 2n\pi + \frac{5\pi}{4} \\ -, \text{ otherwise} \end{cases}$

3.
   (i) $\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}} = \frac{1-\cos A}{\sin A}$, where $A \neq (2n+1)\pi$

   (ii) $\cot\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{1-\cos A}} = \frac{1+\cos A}{\sin A}$, where $A \neq 2n\pi$

See Fig. 2.15.

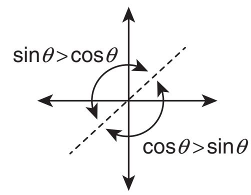

Figure 2.15

The ambiguities of signs are removed by locating the quadrants in which $\frac{A}{2}$ lies or you can follow Fig. 2.16.

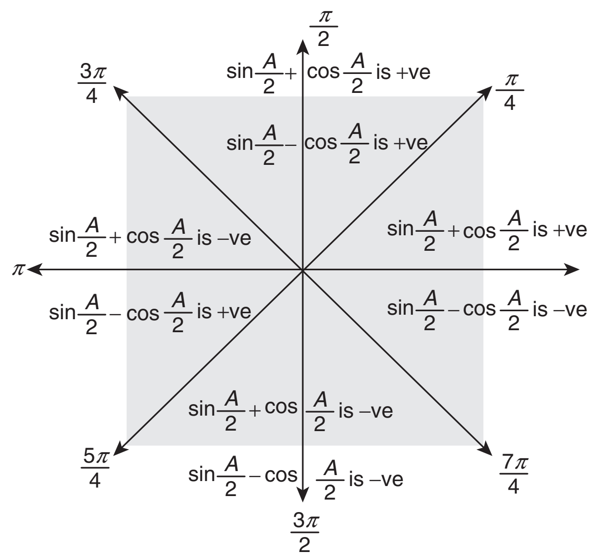

Figure 2.16

4. $\tan^2\frac{A}{2} = \frac{1-\cos A}{1+\cos A}$, where $A \neq (2n+1)\pi$

5. $\cot^2\frac{A}{2} = \frac{1+\cos A}{1-\cos A}$, where $A \neq 2n\pi$

Your Turn 3

1. Show that $\frac{\sin 2A}{1+\cos 2A} = \tan A$.

2. Show that $\frac{\sin A+\sin 2A}{1+\cos A+\cos 2A} = \tan A$.

3. Show that $\sin 3A + \sin 2A - \sin A = 4\sin A\cos\frac{A}{2}\cos\frac{3A}{2}$.

4. Show that $\tan\left(45^\circ+\frac{A}{2}\right) = \sqrt{\frac{1+\sin A}{1-\sin A}}$.

5. If $\tan\theta = \frac{a}{b}$, then show that

$$\frac{\sin\theta}{\cos^8\theta} + \frac{\cos\theta}{\sin^8\theta} = \pm\frac{(a^2+b^2)^4}{\sqrt{a^2+b^2}}\left[\frac{a}{b^8}+\frac{b}{a^8}\right]$$

2.12 Maximum and Minimum Values of $a\cos\theta + b\sin\theta$

Let

$$a = r\cos\alpha \quad (2.6)$$

$$b = r\sin\alpha \quad (2.7)$$

Squaring and adding Eqs. (2.6) and (2.7) we get

$$a^2 + b^2 = r^2 \text{ or } r = \sqrt{a^2+b^2}$$

Therefore,

$$a\sin\theta + b\cos\theta = r(\sin\theta\cos\alpha + \cos\theta\sin\alpha) = r\sin(\theta+\alpha)$$

But $-1 \le \sin\theta \le 1 \Rightarrow -r \le r\sin(\theta+\alpha) \le r$. Hence

$$-\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$$

So the greatest and least values of $a\sin\theta + b\cos\theta$, respectively, are

$$\sqrt{a^2+b^2} \text{ and } -\sqrt{a^2+b^2}$$

Illustration 2.29

1. Prove that $5\cos x + 3\cos(x+\pi/3) + 3$ lies between -4 and 10.
2. Show that, whatever be the value of θ, the expression $a\sin^2\theta + b\sin\theta\cos\theta + c\cos^2\theta$ lies between

$$\frac{(a+c)}{2} - \frac{\sqrt{b^2+(a-c)^2}}{2} \text{ and } \frac{(a+c)}{2} + \frac{\sqrt{b^2+(a-c)^2}}{2}$$

Solution:

1. $5\cos x + 3\left(\cos x\cos\frac{\pi}{3} - \sin x\sin\frac{\pi}{3}\right) + 3$

$$= \left(5+\frac{3}{2}\right)\cos x - \frac{3\sqrt{3}}{2}\sin x + 3$$

$$= \frac{13}{2}\cos x - \frac{3\sqrt{3}}{2}\sin x + 3$$

$$= \sqrt{\frac{169}{4}+\frac{27}{4}}\left\{\frac{13}{2\sqrt{\frac{169}{4}+\frac{27}{4}}}\cos x - \frac{3\sqrt{3}}{2\sqrt{\frac{169}{4}+\frac{27}{4}}}\sin x\right\} + 3$$

$$= 7(\cos\alpha\cos x - \sin\alpha\sin x) + 3 \quad \left[\text{where } \tan\alpha = \frac{3\sqrt{3}}{13}\right]$$

$= 7\cos(\alpha + x) + 3$
$= -1 \le \cos(\alpha + x) \le 1$
$= -7 + 3 \le 7\cos(\alpha + x) + 3 \le 7 + 3$
$= -4 \le 7\cos(\alpha + x) + 3 \le 10$

2. Let $f(\theta) = a\sin^2\theta + b\sin\theta\cos\theta + c\cos^2\theta$. Then

$$f(\theta) = a\frac{(1-\cos 2\theta)}{2} + b\frac{\sin 2\theta}{2} + c\frac{(1+\cos 2\theta)}{2}$$

$$= \frac{1}{2}\left\{(a+c) + \frac{\sqrt{b^2+(a-c)^2}}{2}(\sin 2\theta\cos\alpha - \cos 2\theta\sin\alpha)\right\}$$

$$= \frac{1}{2}\left\{(a+c) + \frac{\sqrt{b^2+(a-c)^2}}{2}\sin(2\theta-\alpha)\right\},\ -1 \le \sin(2\theta-\alpha) \le 1$$

Therefore, $\frac{(a+c)}{2} - \frac{\sqrt{b^2+(a-c)^2}}{2} \le f(\theta) \le \frac{(a+c)}{2} + \frac{\sqrt{b^2+(a-c)^2}}{2}$

Illustration 2.30 The greatest and least values of $\sin x\cos x$ are _____.

Solution:

$$\sin x\cos x = \frac{1}{2}(2\sin x\cos x)$$
$$= \frac{\sin 2x}{2}$$
$$\Rightarrow -1 \le \sin 2x \le 1$$
$$\Rightarrow -\frac{1}{2} \le \frac{\sin 2x}{2} \le \frac{1}{2}$$

Maximum value is 1/2 and minimum value is –1/2

Illustration 2.31 The maximum value of $4\sin^2 x + 3\cos^2 x$ is _____.

Solution:

$$f(x) = 4\sin^2 x + 3\cos^2 x = \sin^2 x + 3 \text{ and } 0 \le |\sin x| \le 1$$

Therefore, maximum value of $4\sin^2 x + 3\cos^2 x$ is 4.

Illustration 2.32 If $A = \cos^2\theta + \sin^4\theta$, then for all values of θ find the range of A.

Solution:

$$A = \cos^2\theta + \sin^4\theta \Rightarrow A = \cos^2\theta + \sin^2\theta\sin^2\theta$$
$$\Rightarrow A \le \cos^2\theta + \sin^2\theta \quad [\because \sin^2\theta \le 1]$$
$$\Rightarrow A \le 1$$

Again

$$A = \cos^2\theta + \sin^4\theta = (1-\sin^2\theta) + \sin^4\theta$$
$$\Rightarrow A = \left(\sin^2\theta - \frac{1}{2}\right)^2 + \frac{3}{4} \ge \frac{3}{4}$$

Hence,

$$\frac{3}{4} \le A \le 1$$

2.13 Trigonometric Series

If we have a cosine series in its product form where the angles are in G.P. with common ratio 2, then multiply both numerator and denominator by 2 sin (least angle).

Illustration 2.33 Simplify the product $\cos A\cos 2A\cos 2^2A \cdots \cos 2^{n-1}A$.

Solution:

$$\cos A\cdot\cos 2A \cdots \cos 2^{n-1}A = \frac{1}{2\sin A}\cdot(\sin A\cdot\cos A)\cdot\cos 2A \cdots \cos 2^{n-1}A$$
$$= \frac{1}{2\sin A}\cdot(\sin 2A\cdot\cos 2A) \cdots \cos 2^{n-1}A$$

$$= \frac{1}{4\sin A}\cdot (\sin 2A\cdot \cos 2A) \cdots \cos 2^{n-1}A$$

Continuing like this we have

$$\cos A\cdot \cos 2A \cdots \cos 2^{n-1}A = \frac{\sin 2^n A}{2^n \sin A}$$

Note:

- $\prod_{r=0}^{n-1} \cos 2^r A = \frac{\sin 2^n A}{2^n \sin A}$ where $\prod$ denotes products.
- If we have a cosine series or a sine series in its sum form where the angles are in AP, then multiply both numerator and denominator with $2\sin\left(\frac{\text{common difference}}{2}\right)$.
- $\sum_{r=1}^{n} \sin[A+(r-1)B] = \frac{\sin\left[A+\frac{(n-1)}{2}B\right]\sin\frac{nB}{2}}{\sin\frac{B}{2}}$.
- $\sum_{r=1}^{n} \cos[A+(r-1)B] = \frac{\cos\left[A+\frac{(n-1)}{2}B\right]\sin\frac{nB}{2}}{\sin\frac{B}{2}}$, where $\sum$ denotes summation.

Illustration 2.34 Prove that $\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7} = -\frac{1}{2}$.

Solution:

$$\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7} = \frac{2\sin\frac{\pi}{7}}{2\sin\frac{\pi}{7}}\left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}\right)$$

$$= \frac{\left(\sin\frac{3\pi}{7}-\sin\frac{\pi}{7}+\sin\frac{5\pi}{7}-\sin\frac{3\pi}{7}+\sin\pi-\sin\frac{5\pi}{7}\right)}{2\sin\frac{\pi}{7}} = -\frac{1}{2}$$

Illustration 2.35 Sum to n-terms of the series

$$\sin\alpha - \sin(\alpha+\beta)+\sin(\alpha+2\beta)-\sin(\alpha+3\beta)+\cdots$$

Solution:

Since,

$$\sin(\pi+\alpha) = -\sin\alpha \text{ and } \sin(2\pi+\alpha) = \sin\alpha$$

Therefore,

$$-\sin(\alpha+\beta) = \sin(\pi+\alpha+\beta)$$
$$\sin(\alpha+2\beta) = \sin(2\pi+\alpha+2\beta)$$
$$-\sin(\alpha+3\beta) = \sin(3\pi+\alpha+3\beta) \text{ and so on.}$$

Using these results, the required sum is

$$S = \sin\alpha + \sin(\pi+\alpha+\beta)+\sin(2\pi+\alpha+2\beta)+\sin(3\pi+\alpha+3\beta)+\cdots$$
upto n terms

$$S = \frac{\sin n\frac{\pi+\beta}{2}}{\sin\frac{\pi+\beta}{2}}\cdot \sin\left[\alpha+(n-1)\frac{\pi+\beta}{2}\right]$$

Illustration 2.36 Show that $\cos\frac{\pi}{11}\cos\frac{2\pi}{11}\cos\frac{3\pi}{11}\cos\frac{4\pi}{11}\cos\frac{5\pi}{11} = \frac{1}{32}$.

Solution:

$$\cos\frac{\pi}{11}\cos\frac{2\pi}{11}\cos\frac{3\pi}{11}\cos\frac{4\pi}{11}\cos\frac{5\pi}{11} = \frac{1}{32}$$

$$\text{LHS} = \cos\frac{\pi}{11}\cos\frac{2\pi}{11}\cos\frac{3\pi}{11}\cos\frac{4\pi}{11}\cos\frac{5\pi}{11}$$

Let $\frac{\pi}{11} = \alpha$. Then the above equation can be written as

$$\cos\alpha\cos 2\alpha\cos 3\alpha\cos 4\alpha\cos 5\alpha$$
$$= -\cos\alpha\cos 2\alpha\cos 4\alpha\cos 8\alpha\cos 5\alpha \; (11\alpha=\pi \Rightarrow 3\alpha = \pi - 8\alpha)$$
$$= -\cos 2^0\alpha\cos 2^1\alpha\cos 2^2\alpha\cos 2^3\alpha\cos 5\alpha$$

Using formula $\cos\alpha\cos 2\alpha \; \cos 4\alpha - \cos 2^{n-1}\alpha = \frac{\sin 2^n\alpha}{2^n\sin\alpha}$

$$= -\frac{\sin 2^4}{2^4\sin\alpha}\cos 5\alpha = -\frac{\sin\frac{16\pi}{11}\cos\frac{5\pi}{11}}{16\sin\frac{\pi}{11}}$$

$$= \frac{2\sin\frac{5\pi}{11}\cos\frac{5\pi}{11}}{32\sin\frac{\pi}{11}} = \frac{1}{32}$$

2.14 Conditional Trigonometrical Identities

1. **Identities:** A trigonometric equation is an identity if it is true for all values of the angle or angles involved.
2. **Conditional identities:** When the angles involved satisfy a given relation, the identity is called conditional identity. In proving these identities we require properties of complementary and supplementary angles.

2.14.1 Important Conditional Identities

(A) If $A+B+C=\pi$, then

1. $\sin 2A+\sin 2B+\sin 2C = 4\sin A\sin B\sin C$
2. $\sin 2A+\sin 2B-\sin 2C = 4\cos A\cos B\sin C$
3. $\sin(B+C-A)+\sin(A+C-B)+\sin(B+A-C)$ $= 4\sin A\sin B\sin C$
4. $\cos 2A+\cos 2B+\cos 2C = -1-4\cos A\cos B\cos C$
5. $\cos 2A+\cos 2B-\cos 2C = 1-4\sin A\sin B\cos C$
6. $\sin A+\sin B+\sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$
7. $\sin A+\sin B-\sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$
8. $\cos A+\cos B+\cos C = 1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
9. $\cos A+\cos B-\cos C = -1+4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$
10. $\frac{\cos A}{\sin B\sin C}+\frac{\cos B}{\sin A\sin C}+\frac{\cos C}{\sin B\sin A} = 2$

11. $\sin^2 A+\sin^2 B-\sin^2 C=2\sin A\sin B\cos C$

12. $\cos^2 A+\cos^2 B+\cos^2 C=1-2\cos A\cos B\cos C$

13. $\sin^2 A+\sin^2 B+\sin^2 C=2+2\cos A\cos B\cos C$

14. $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

15. $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=2+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

16. $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}=1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

17. $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}-\cos^2\frac{C}{2}=2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

18. $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

19. $\cot A\cot B+\cot B\cot C+\cot C\cot A=1$

20. $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$

21. $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2}$

(B) If $x+y+z=\frac{\pi}{2}$, then

22. $\sin^2 x+\sin^2 y+\sin^2 z=1-2\sin x\cdot\sin y\cdot\sin z$

23. $\cos^2 x+\cos^2 y+\cos^2 z=2+2\sin x\cdot\sin y\cdot\sin z$

24. $\sin 2x+\sin 2y+\sin 2z=4\cos x\cdot\cos y\cdot\cos z$

Illustration 2.37 If $A+B+C=\pi$, then $\cos^2 A+\cos^2 B-\cos^2 C$ is equal to ______.

Solution:

$$\cos^2 A+\cos^2 B-\cos^2 C=\cos^2 A+1-\sin^2 B-\cos^2 C$$
$$=1+\cos^2 A-\sin^2 B-\cos^2 C=1+\cos(A+B)\cos(A-B)-\cos^2 C$$
$$=1+\cos(\pi-C)\cos(A-B)-\cos^2 C=1-\cos C\cos(A-B)-\cos^2 C$$
$$=1-\cos C[\cos(A-B)+\cos C]=1-\cos C[\cos(A-B)-\cos(A+B)]$$
$$=1-2\sin A\sin B\cos C$$

Illustration 2.38 If $A+B+C=\pi$, then the value of $(\cot A+\cot B)$ $(\cot C+\cot B)$ $(\cot A+\cot C)$ will be _____.

Solution:

$$\cot A+\cot B=\frac{\sin A\cos B+\sin B\cos A}{\sin A\sin B}=\frac{\sin(A+B)}{\sin A\sin B}=\frac{\sin C}{\sin A\sin B}$$

Similarly,

$$\cot C+\cot B=\frac{\sin A}{\sin C\sin B}$$

and

$$\cot C+\cot A=\frac{\sin B}{\sin C\sin A}$$

Therefore,

$$(\cot A+\cot B)(\cot C+\cot B)(\cot A+\cot C)$$
$$=\frac{\sin C}{\sin A\sin B}\cdot\frac{\sin A}{\sin C\sin B}\cdot\frac{\sin B}{\sin C\sin A}=\operatorname{cosec} A\cdot\operatorname{cosec} B\cdot\operatorname{cosec} C$$

Illustration 2.39 If A,B and C are angles of a triangle, then $\sin 2A+\sin 2B-\sin 2C$ is equal to ______.

Solution:

$$\sin 2A+\sin 2B-\sin 2C=2\sin A\cos A+2\cos(B+C)\sin(B-C)\quad(1)$$

Since, $A + B + C = \pi$. We have $B+C=\pi-A$. Hence

$$\cos(B+C)=-\cos A \text{ and } \sin(B+C)=\sin A$$

Taking the RHS of Eq. (1) and substituting $\cos(B + C) = -\cos A$, $\sin(B + C) = \sin A$ we get

$$2\cos A[\sin A-\sin(B-C)]=2\cos A[\sin(B+C)-\sin(B-C)]$$
$$=4\cos A\cos B\sin C$$

Illustration 2.40 If $x+y+z=xyz$, prove that

$$\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}=\frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$$

Solution:

Let $x = \tan A$, $y = \tan B$, $z = \tan C$. Therefore

$$\tan A + \tan B + \tan C = \tan A\cdot\tan B\cdot\tan C$$
$$\Rightarrow A + B + C = \pi$$

Hence,

$$\tan(2A + 2B) = \tan(2\pi - 2C)$$
$$\Rightarrow \tan(2A + 2B) = -\tan 2C$$
$$\Rightarrow \tan 2A + \tan 2B + \tan 2C = \tan 2A\cdot\tan 2B\cdot\tan 2C$$
$$\Rightarrow \frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{8xyz}{(1-x^2)(1-y^2)(1-z^2)}$$
$$\left(\tan 2A=\frac{2\tan A}{1-\tan^2 A}\right)$$
$$\Rightarrow \frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}=\frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$$

Illustration 2.41 If $A + B + C = 180°$, prove that

$$\sin(B + C - A) + \sin(C + A - B) + \sin(A + B - C) = 4\sin A\sin B\sin C$$

Solution:

$$\text{LHS} = \sin(B + C - A) + \sin(C + A - B) + \sin(A + B - C)$$
$$= \sin(\pi - A - A) + \sin(\pi - B - B) + \sin(\pi - C - C)\quad(\because A+B+C=\pi)$$
$$= \sin 2A + \sin 2B + \sin 2C$$
$$= 4\sin A\sin B\sin C$$

Illustration 2.42 If in ΔABC, $\cos^3 A + \cos^3 B + \cos^3 C = 3\cos A\cos B\cos C$, then prove that the triangle is equilateral.

Solution:

Given that $\cos^3 A + \cos^3 B + \cos^3 C - 3\cos A\cos B\cos C = 0$. So

$$(\cos A + \cos B + \cos C)(\cos^2 A + \cos^2 B + \cos^2 C - \cos A\cos B - \cos B\cos C - \cos C\cos A) = 0$$
$$\Rightarrow \cos^2 A + \cos^2 B + \cos^2 C - \cos A\cos B - \cos B\cos C - \cos C\cos A = 0$$

(as $\cos A + \cos B + \cos C = 1 + 4 \sin A/2 \sin B/2 \sin C/2 \neq 0$)

$\Rightarrow (\cos A - \cos B)^2 + (\cos B - \cos C)^2 + (\cos C - \cos A)^2 = 0$

$\Rightarrow \cos A = \cos B = \cos C$

$\Rightarrow A = B = C, (\because 0 < A, B, C < \pi)$

$\Rightarrow \Delta ABC$ is equilateral.

Illustration 2.43 If A, B and C are angles of a triangle, prove that

$$E = \frac{\cos\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B+C}{2}\right)} + \frac{\cos\left(\frac{C-A}{2}\right)}{\cos\left(\frac{C+A}{2}\right)} + \frac{\cos\left(\frac{A-B}{2}\right)}{\cos\left(\frac{A+B}{2}\right)}$$

Solution:

Since $A + B + C = \pi$

$$E = \frac{\cos\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B+C}{2}\right)} + \frac{\cos\left(\frac{C-A}{2}\right)}{\cos\left(\frac{C+A}{2}\right)} + \frac{\cos\left(\frac{A-B}{2}\right)}{\cos\left(\frac{A+B}{2}\right)}$$

$$= \frac{\cos\left(\frac{B-C}{2}\right)}{\sin\left(\frac{A}{2}\right)} + \frac{\cos\left(\frac{C-A}{2}\right)}{\sin\left(\frac{B}{2}\right)} + \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C}{2}\right)}$$

$$= \frac{2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)}{\sin A} + \frac{2\cos\left(\frac{B}{2}\right)\cos\left(\frac{C-A}{2}\right)}{\sin B} + \frac{2\cos\left(\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right)}{\sin C}$$

$$= \frac{\sin B + \sin C}{\sin A} + \frac{\sin A + \sin C}{\sin B} + \frac{\sin B + \sin A}{\sin C}$$

$$= \left(\frac{\sin B}{\sin A} + \frac{\sin A}{\sin B}\right) + \left(\frac{\sin C}{\sin B} + \frac{\sin B}{\sin C}\right) + \left(\frac{\sin C}{\sin A} + \frac{\sin A}{\sin C}\right)$$

As A, B, C are angles of a triangle, hence $0 < A, B, C < \pi$. So $\sin A, \sin B, \sin C > 0$

$$\Rightarrow E \geq 2 + 2 + 2 \left(\text{as } x + \frac{1}{x} \geq 2 \text{ if } x > 0\right)$$

2.15 Height and Distance

Angle of elevation: Let 'O' be the observer's eye and OA be the horizontal line through O. If the object B is at a higher level than eye, then angle $AOB = \theta$ is called the angle of elevation (Fig. 2.17).

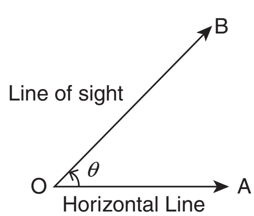

Figure 2.17

Angle of depression: If the object B is at a lower level than O, then angle AOB is called the angle of depression (Fig. 2.18).

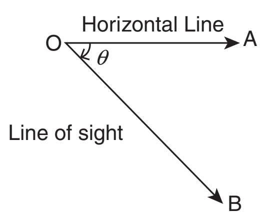

Figure 2.18

Illustration 2.44 A man is standing away from a tower of 150 meter height. At the top of the tower, the angle of depression of the man changes from 60° to 45° when man moves towards the tower. Find the distance travelled by the man.

Solution:

Let PQ be the tower, where $PQ = 150$ meters (Fig. 2.19). Let the two positions of the man be at A and B so that

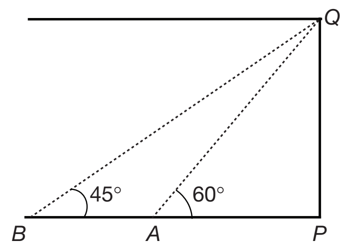

Figure 2.19

$$AP = h \cot 60° = 150\sqrt{3}$$
$$BP = 150 \cot 45° = 150$$

Now

$$AB = 150 - \frac{150}{\sqrt{3}} = \frac{150}{\sqrt{3}}(\sqrt{3} - 1)$$

Hence, the distance traveled by the man is $\frac{150}{\sqrt{3}}(\sqrt{3} - 1)$ m.

Additional Solved Examples

1. $\sqrt{3}$ cosec 20° – sec 20° = _____.

(A) 1 **(B)** 2

(C) 3 **(D)** 4

Solution:

$$\frac{4 \cdot \left(\frac{\sqrt{3}}{2}\cos 20° - \frac{1}{2}\sin 20°\right)}{2\sin 20° \cos 20°} = 4 \cdot \frac{\sin 60° \cos 20° - \cos 60° \sin 20°}{\sin 40°}$$
$$= 4$$

Hence, the correct answer is option (D).

2. $\tan A + 2 \tan 2A + 4 \tan 4A + 8 \cot 8A =$ _____.

(A) $\cot A$ **(B)** $\tan 6A$

(C) $\cot 4A$ **(D)** None of these

Solution:

$$\tan A + 2 \tan 2A + 4 \tan 4A + 8 \cot 8A$$
$$= \tan A + 2\tan 2A + 4\tan 4A + 8\,\frac{1 - \tan^2 4A}{2\tan 4A}$$
$$= \tan A + 2\tan 2A + 4\cot 4A = \tan A + 2\tan 2A + 4\,\frac{1 - \tan^2 2A}{2\tan 2A}$$
$$= \tan A + 2\cot 2A = \tan A + \frac{1 - \tan^2 A}{\tan A} = \cot A$$

Hence, the correct answer is option (A).

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