ST. LAWRENCE HIGH SCHOOL

A JESUIT CHRISTIAN MINORITY INSTITUTION

FOR GOD AND COUNTRY

STUDY MATERIAL-16

SUBJECT – MATHEMATICS

Pre-Test

Chapter: Integration **Class: XII**

Topic: Indefinite Integral **Date: 25.06.2020**

-:Indefinite Integral:-

22.1 Primitive or Anti-Derivative of a Function

A function $\phi(x)$ is called a **primitive** or an **anti-derivative** of a function $f(x)$ if $\phi'(x)=f(x)$.

For example, $\frac{x^5}{5}$ is a primitive of x^4, because $\frac{d}{dx}\left(\frac{x^5}{5}\right)=x^4$.

Let $\phi(x)$ be a primitive of a function $f(x)$ and let c be any constant. Then

$$\frac{d}{dx}(\phi(x)+c)=\phi'(x)=f(x) \text{ [since } \phi'(x)=f(x)]$$

So, $\phi(x)+c$ is also a primitive of $f(x)$.

Thus, if a function $f(x)$ possesses a primitive, then it possesses infinitely many primitives that are contained in the expression $\phi(x)+c$ where c is a constant.

For example, $\frac{x^5}{5}, \frac{x^5}{5}-2, \frac{x^5}{5}+1$, etc. are primitives of x^4.

22.2 Indefinite Integral and Indefinite Integration

Let $f(x)$ be a function. Then the collection of all its primitives is called the **indefinite integral** of $f(x)$ and is denoted by $\int f(x)dx$.

Thus, $\frac{d}{dx}(\phi(x)+c)=f(x)\Rightarrow\int f(x)dx=\phi(x)+c$.

where $\phi(x)$ is the primitive of $f(x)$ and c is an arbitrary constant known as the **constant of integration**.

Here $\int$ is the integral sign, $f(x)$ is the integrand, x is the variable of integration and dx is the element of integration.

The process of finding an indefinite integral of a given function is called integration of the function.

It follows from the above discussion that integrating a function $f(x)$ means finding a function $\phi(x)$ such that $\frac{d}{dx}\phi(x)=f(x)$.

22.2.1 Fundamental Properties of Integration

1. $\int c\cdot f(x)dx=c\int f(x)dx$
2. $\int (f(x)\pm g(x))dx=\int f(x)dx\pm\int g(x)dx$
3. $\int f(x)dx=g(x)+c\Rightarrow\int f(ax+b)dx=\frac{1}{a}g(ax+b)+c$

Note: Every continuous function is integrable.

22.2.2 Fundamental Formulas on Integration

1. $\int 1dx=x+c$
2. $\int x^n\,dx=\frac{x^{n+1}}{n+1}+c,\ n\neq -1$ $\left[\text{since, } \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)=x^n\right]$
3. $\int \frac{1}{x}dx=\ln(x)+c\left[\text{since, } \frac{d}{dx}(\ln x)=\frac{1}{x}\right]$
4. $\int (ax+b)^n\,dx=\frac{1}{a}\cdot\frac{(ax+b)^{n+1}}{n+1}+c,\ n\neq -1$
5. $\int \frac{1}{(ax+b)}dx=\frac{1}{a}\cdot\ln|ax+b|+c$
6. $\int e^x\,dx=e^x+c$ $\left[\text{since, } \frac{d}{dx}(e^x)=e^x\right]$
7. $\int a^x\,dx=\frac{a^x}{\ln a}+c$ $\left[\text{since, } \frac{d}{dx}\left(\frac{a^x}{\ln a}\right)=a^x\right]$
8. $\int \sin x\,dx=-\cos x+c$ $\left[\text{since, } \frac{d}{dx}(-\cos x)=\sin x\right]$
9. $\int \cos x\,dx=\sin x+c$ $\left[\text{since, } \frac{d}{dx}(\sin x)=\cos x\right]$
10. $\int \sec^2 x\,dx=\tan x+c$ $\left[\text{since, } \frac{d}{dx}(\tan x)=\sec^2 x\right]$
11. $\int \text{cosec}^2 x\,dx=-\cot x+c$ $\left[\text{since, } \frac{d}{dx}(-\cot x)=\text{cosec}^2 x\right]$
12. $\int \sec x\cdot\tan x\,dx=\sec x+c$ $\left[\text{since, } \frac{d}{dx}(\sec x)=\sec x\cdot\tan x\right]$
13. $\int \text{cosec}\,x\cdot\cot x\,dx=-\text{cosec}\,x+c$

    $\left[\text{since, } \frac{d}{dx}(-\text{cosec}\,x)=\text{cosec}\,x\cdot\cot x\right]$
14. $\int \tan x\,dx=\ln|\sec x|+c=-\ln|\cos x|+c$

    $\left[\text{since, } \frac{d}{dx}(\ln\cos x)=-\tan x\right]$
15. $\int \cot x\,dx=\ln|\sin x|+c=-\ln|\text{cosec}\,x|+c$

    $\left[\text{since, } \frac{d}{dx}(\ln\sin x)=\cot x\right]$

16. $\int \sec x\,dx = \ln|\sec x + \tan x| + c = \ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)+c$

$\left[\text{since, } \frac{d}{dx}(\ln(\sec x+\tan x)) = \sec x\right]$

17. $\int \operatorname{cosec} x\,dx = \ln|\operatorname{cosec} x - \cot x| + c = \ln\tan\left(\frac{x}{2}\right)+c$

$\left[\text{since, } \frac{d}{dx}(\ln(\operatorname{cosec} x-\cot x)) = \operatorname{cosec} x\right]$

18. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + c = -\cos^{-1}x + c$

$\left[\text{since, } \frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}\right]$

19. $\int \frac{dx}{1+x^2} = \tan^{-1}x + c = -\cot^{-1}x + c$

$\left[\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}, \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}\right]$

20. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x + c = -\operatorname{cosec}^{-1}x + c$

$\left[\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}, \frac{d}{dx}(\operatorname{cosec}^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}\right]$

22.2.2.1 Some Standard Results on Integration

21. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c = -\cos^{-1}\frac{x}{a} + c$

$\left[\text{since } \frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{a^2-x^2}}\right]$

22. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\cdot\tan^{-1}\frac{x}{a} + c = -\frac{1}{a}\cdot\cot^{-1}\frac{x}{a} + c$

$\left[\text{since } \frac{d}{dx}\left(\tan^{-1}\frac{x}{a}\right) = \frac{a}{a^2+x^2}\right]$

23. $\int \frac{dx}{x\cdot\sqrt{x^2-a^2}} = \frac{1}{a}\cdot\sec^{-1}\frac{x}{a} + c = -\frac{1}{a}\cdot\operatorname{cosec}^{-1}\frac{x}{a} + c$

$\left[\text{since } \frac{d}{dx}\left(\sec^{-1}\frac{x}{a}\right) = \frac{a}{x\cdot\sqrt{x^2-a^2}}\right]$

24. $\int \frac{dx}{x^2-a^2} = -\frac{1}{a}\cdot\cot h^{-1}\frac{x}{a} + c = \frac{1}{2a}\cdot\ln\left|\frac{x-a}{x+a}\right| + c,\ x > a$

25. $\int \frac{dx}{a^2-x^2} = -\frac{1}{a}\cdot\tan h^{-1}\frac{x}{a} + c = \frac{1}{2a}\cdot\ln\left|\frac{a+x}{a-x}\right| + c,\ x < a$

26. $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right| + c = \cos h^{-1}\frac{x}{a} + c$

27. $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x+\sqrt{x^2+a^2}\right| + c = \sin h^{-1}\frac{x}{a} + c$

28. $\int \sqrt{a^2-x^2}\ dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

29. $\int \sqrt{x^2-a^2}\ dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + c$

30. $\int \sqrt{x^2+a^2}\ dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + c$

Key points:

1. The signum function has an anti-derivative on any interval which does not contain the point $x = 0$, and does not possess an anti-derivative on any interval which contains the point.
2. The anti-derivative of every odd function is an even function and vice versa.

Illustration 22.1 Evaluate $\int \frac{\sin x}{1+\sin x}dx$.

Solution:

$$\int \frac{\sin x}{1+\sin x}dx = \int \frac{\sin x}{(1+\sin x)}\cdot\frac{(1-\sin x)}{(1-\sin x)}dx = \int \frac{\sin x - \sin^2 x}{\cos^2 x}dx$$

$$= \int\left(\sec x\cdot\tan x - \tan^2 x\right)dx$$

$$= \int(1-\sec^2 x + \sec x\cdot\tan x)dx = x - \tan x + \sec x + c$$

Illustration 22.2 Evaluate $\int \frac{(x+1)^2}{x(x^2+1)}dx$.

Solution:

$$\int \frac{(x+1)^2}{x(x^2+1)}dx = \int \frac{(x^2+2x+1)}{x(x^2+1)}dx$$

$$= \int \frac{x^2+1}{x(x^2+1)}dx + \int \frac{2x}{x(x^2+1)}dx$$

$$= \int \frac{1}{x}dx + \int \frac{2}{(x^2+1)}dx = \ln x + 2\tan^{-1}x + c$$

Illustration 22.3 Evaluate $\int \frac{ax^3+bx^2+c}{x^4}dx$.

Solution:

$$\int \frac{ax^3+bx^2+c}{x^4}dx = \int\left(\frac{a}{x}+\frac{b}{x^2}+\frac{c}{x^4}\right)dx = a\ln x - \frac{b}{x} - \frac{c}{3x^3} + k$$

Illustration 22.4 Evaluate $\int \frac{1+x+\sqrt{x+x^2}}{\sqrt{x}+\sqrt{1+x}}dx$.

Solution:

$$\int \frac{1+x+\sqrt{x+x^2}}{\sqrt{x}+\sqrt{x+1}}dx = \int \frac{\sqrt{x+1}(\sqrt{x}+\sqrt{x+1})}{\sqrt{x}+\sqrt{x+1}}dx$$

$$= \int \sqrt{x+1}\,dx = \frac{2}{3}(x+1)^{\frac{3}{2}} + c$$

Illustration 22.5 Evaluate $\int(\sin^4 x - \cos^4 x)dx$.

Solution:

$$\int(\sin^4 x - \cos^4 x)dx = \int(\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)dx$$

$$= \int(\sin^2 x - \cos^2 x)\ dx = -\int \cos 2x\,dx = -\frac{1}{2}\sin 2x + c$$

Illustration 22.6 Evaluate $\int \sqrt{1+\sin\left(\frac{x}{4}\right)}\,dx$.

Solution:

$$\int \sqrt{1+\sin\left(\frac{x}{4}\right)}\,dx = \int \sqrt{\sin^2\left(\frac{x}{8}\right)+\cos^2\left(\frac{x}{8}\right)+2\sin\left(\frac{x}{8}\right)\cos\left(\frac{x}{8}\right)}\,dx$$

$$= \int \sqrt{\left[\sin\left(\frac{x}{8}\right)+\cos\left(\frac{x}{8}\right)\right]^2}\,dx$$

$$= \int \left[\sin\left(\frac{x}{8}\right)+\cos\left(\frac{x}{8}\right)\right]dx = -\frac{\cos(x/8)}{1/8}+\frac{\sin(x/8)}{1/8}+c$$

$$= 8[\sin(x/8)-\cos(x/8)]+c$$

Illustration 22.7 Evaluate $\int \frac{2x}{(2x+1)^2}\,dx$.

Solution:

$$\int \frac{2x}{(2x+1)^2}\,dx = \int \frac{2x+1-1}{(2x+1)^2}\,dx$$

$$= \int \frac{1}{(2x+1)}\,dx - \int \frac{1}{(2x+1)^2}\,dx$$

$$= \frac{1}{2}\ln|2x+1| + \frac{1}{2(2x+1)} + c$$

Illustration 22.8 Evaluate $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cdot \cos^2 x}\,dx$.

Solution:

$$\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cdot \cos^2 x}\,dx = \int \sec^2 x\,dx - \int \operatorname{cosec}^2 x\,dx$$

$$= \tan x + \cot x + c$$

Illustration 22.9 Evaluate $\int (3\operatorname{cosec}^2 x + 2\sin 3x)\,dx$.

Solution:

$$\int (3\operatorname{cosec}^2 x + 2\sin 3x)\,dx = -3\cot x - \frac{2}{3}\cos 3x + c$$

Illustration 22.10 Evaluate $\int \frac{1}{\sqrt{1+x}+\sqrt{x}}\,dx$.

Solution:

$$\int \frac{1}{\sqrt{1+x}+\sqrt{x}}\,dx = \int \frac{(\sqrt{1+x}-\sqrt{x})}{(\sqrt{1+x}+\sqrt{x})\cdot(\sqrt{1+x}-\sqrt{x})}\,dx$$

$$= \int (\sqrt{1+x}-\sqrt{x})\,dx = \frac{(x+1)^{3/2}}{3/2} - \frac{(x)^{3/2}}{3/2} + c$$

$$= \frac{2}{3}[(x+1)^{3/2}-(x)^{3/2}]+c$$

Your Turn 1

1. $\int \frac{\sin x}{\sin(x-\alpha)}\,dx =$

(A) $x\cos\alpha - \sin\alpha \ln\sin(x-\alpha)+c$

(B) $x\cos\alpha + \sin\alpha \ln\sin(x-\alpha)+c$

(C) $x\sin x - \alpha - \sin\alpha \ln\sin(x-\alpha)+c$

(D) None of these

Ans. (B)

2. $\int \frac{\cos x - 1}{\cos x + 1}\,dx =$

(A) $2\tan\frac{x}{2} - x + c$ **(B)** $\frac{1}{2}\tan\frac{x}{2} - x + c$

(C) $-\frac{1}{2}\tan\frac{x}{2} + x + c$ **(D)** $-2\tan\frac{x}{2} + x + c$

Ans. (D)

3. $\int \frac{1}{1-\sin x}\,dx =$

(A) $x+\cos x+c$ **(B)** $1+\sin x+c$

(C) $\sec x - \tan x + c$ **(D)** $\sec x + \tan x + c$

Ans. (D)

4. If $\int (\sin 2x - \cos 2x)\,dx = \frac{1}{\sqrt{2}}\sin(2x-\alpha)+b$, then

(A) $a = \frac{\pi}{4}, b = 0$

(B) $a = -\frac{\pi}{4}, b = 0$

(C) $a = \frac{5\pi}{4}$, $b =$ any constant

(D) $a = -\frac{5\pi}{4}$, $b =$ any constant **Ans.** (D)

5. $\int \left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\right)dx =$

(A) $-e^x + c$ **(B)** $e^x + c$

(C) $e^{-x}+c$ **(D)** $-e^{-x}+c$

Ans. (B)

6. $\int \frac{\cot x \cdot \tan x}{\sec^2 x - 1}\,dx =$

(A) $\cot x - x + c$ **(B)** $-\cot x + x + c$

(C) $\cot x + x + c$ **(D)** $-\cot x - x + c$

Ans. (D)

7. $\int (\sec x + \tan x)^2\,dx =$

(A) $2(\sec x + \tan x) - x + c$ **(B)** $\frac{1}{3}(\sec x + \tan x)^3 + c$

(C) $\sec x(\sec x + \tan x) + c$ **(D)** $2(\sec x + \tan x) + c$

Ans. (A)

8. $\int x^{51}(\tan^{-1}x + \cot^{-1}x)\,dx =$

(A) $\frac{x^{52}}{52}(\tan^{-1}x + \cot^{-1}x)+c$ **(B)** $\frac{x^{52}}{52}(\tan^{-1}x - \cot^{-1}x)+c$

(C) $\frac{\pi x^{52}}{104}+\frac{\pi}{2}+c$ **(D)** $\frac{\pi x^{52}}{52}+\frac{\pi}{2}+c$ **Ans.** (A)

9. $\int 5\sin x\,dx =$

(A) $5\cos x + c$ **(B)** $-5\cos x + c$

(C) $5\sin x + c$ **(D)** $-5\sin x + c$

Ans. (B)

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