ST. LAWRENCE HIGH SCHOOL
A JESUIT CHRISTIAN MINORITY INSTITUTION
FOR GOD AND COUNTRY

STUDY MATERIAL-18

SUBJECT – MATHEMATICS

Pre-Test

Chapter: Integration **Class: XII**

Topic: Method Of Substitution **Date: 27.06.2020**

–:Method of Substitution:–

5. Standard substitutions:-

	Integrand form	Substitution
(a)	$\sqrt{a^2-x^2}$, $\dfrac{1}{\sqrt{a^2-x^2}}$, a^2-x^2	$x=a\sin\theta$ or $a\cos\theta$
(b)	$\sqrt{a^2+x^2}$, $\dfrac{1}{\sqrt{a^2+x^2}}$, a^2+x^2	$x=a\tan\theta$ or $a\cot\theta$
(c)	$\sqrt{x^2-a^2}$, $\dfrac{1}{\sqrt{x^2-a^2}}$, x^2-a^2	$x=a\sec\theta$ or $a\operatorname{cosec}\theta$
(d)	$\sqrt{\dfrac{x}{x+a}}$, $\sqrt{\dfrac{x+a}{x}}$, $\sqrt{x(x+a)}$, $\sqrt{\dfrac{1}{x(x+a)}}$	$x=a\tan^2\theta$ or $a\cot^2\theta$
(e)	$\sqrt{\dfrac{x}{a-x}}$, $\sqrt{\dfrac{a-x}{x}}$, $\sqrt{x(a-x)}$, $\dfrac{1}{\sqrt{x(a-x)}}$	$x=a\sin^2\theta$ or $x=a\cos^2\theta$
(f)	$\sqrt{\dfrac{x}{x-a}}$, $\sqrt{\dfrac{x-a}{x}}$, $\sqrt{x(x-a)}$, $\sqrt{\dfrac{1}{x(x-a)}}$	$x=a\sec^2\theta$ or $a\operatorname{cosec}^2\theta$
(g)	$\sqrt{\dfrac{a+x}{a-x}}$, $\sqrt{\dfrac{a-x}{a+x}}$	$x=a\cos 2\theta$
(h)	$\sqrt{\dfrac{x-\alpha}{\beta-x}}$, $\sqrt{(x-\alpha)(\beta-x)}$, $(\beta>\alpha)$	$x=\beta\sin^2\theta+\alpha\cos^2\theta$

Solved Examples :-

Example 1 Evaluate $\int \frac{dx}{(1+\sqrt{x})\sqrt{x-x^2}}$.

Solution: Let $x=\sin^2\theta$. Then

$$dx = 2\sin\theta\cdot\cos\theta\cdot d\theta$$

$$I=\int \frac{2\sin\theta\cdot\cos\theta\cdot d\theta}{(1+\sin\theta)\sqrt{\sin^2\theta-\sin^4\theta}}=\int\frac{2d\theta}{(1+\sin\theta)}=\int\frac{2(1-\sin\theta)d\theta}{\cos^2\theta}$$

$$\Rightarrow I=2\int(\sec^2\theta-\sec\theta\cdot\tan\theta)d\theta=2(\tan\theta-\sec\theta)+c$$

$$\Rightarrow I=2\left[\frac{\sin\theta-1}{\cos\theta}\right]+c=\frac{2\left(\sqrt{x}-1\right)}{\sqrt{1-x}}+c$$

Example 2 Evaluate $\int\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$.

Solution:

$$I=\int\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx=\int\sqrt{\frac{(1-\sqrt{x})^2}{1-x}}\,dx=\int\frac{1}{\sqrt{1-x}}dx-\int\frac{\sqrt{x}}{\sqrt{1-x}}\,dx$$

Let $x = \sin^2\theta$. Then

$$dx = 2\sin\theta \cdot \cos\theta \cdot d\theta$$

$$I = \int 2\sin\theta\, d\theta - \int 2\sin^2\theta\, d\theta = 2\int \sin\theta\, d\theta - \int (1-\cos 2\theta)\, d\theta$$

$$= -2\cos\theta - \theta + \frac{\sin 2\theta}{2} + c$$

$$I = \sqrt{x} \cdot \sqrt{1-x} - 2\sqrt{1-x} - \sin^{-1}\sqrt{x} + c$$

Example 3 Evaluate $\int t\sqrt{\dfrac{t^2+1}{t^2-1}}dt$.

Solution: Put $s = t^2$. Then $ds = 2tdt$.

Now,

$$I = \frac{1}{2}\int \sqrt{\frac{s+1}{s-1}}\, ds = \frac{1}{2}\int \frac{1+s}{\sqrt{s^2-1}}\, ds$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{s^2-1}}\, ds + \frac{1}{2}\int \frac{s}{\sqrt{s^2-1}}\, ds = \frac{1}{2}\ln\left|s+\sqrt{s^2-1}\right| + \frac{1}{4}\int \frac{2sds}{\sqrt{s^2-1}}$$

Let $s^2 = x \Rightarrow 2sds = dx$. Then

$$\frac{1}{4}\int \frac{2sds}{\sqrt{s^2-1}} = \frac{1}{4}\int \frac{dx}{\sqrt{x-1}} = \frac{1}{2}\left(\sqrt{x-1}\right) = \frac{1}{2}\sqrt{s^2-1}$$

So,

$$I = \frac{1}{2}\ln\left|s+\sqrt{s^2-1}\right| + \frac{1}{2}\sqrt{s^2-1}$$

$$= \frac{1}{2}\ln\left|t^2+\sqrt{t^4-1}\right| + \frac{1}{2}\sqrt{t^4-1} + c$$

Example 4 Evaluate $\int \frac{dx}{(a^2+x^2)^{3/2}}$.

Solution:

$$I = \int \frac{dx}{(a^2+x^2)^{3/2}}$$

Put $x = a\tan\theta$. Then

$$dx = a\sec^2\theta\, d\theta$$

Therefore,

$$I = \int \frac{a\sec^2\theta\, d\theta}{(a^2+(a\tan\theta)^2)^{3/2}}$$

$$\Rightarrow I=\int \frac{a\sec^2\theta\, d\theta}{a^3\sec^3\theta}=\int \frac{d\theta}{a^2\sec\theta}$$

$$\Rightarrow I=\frac{1}{a^2}\int \frac{d\theta}{\sec\theta}=\frac{1}{a^2}\int \cos\theta\, d\theta = \frac{1}{a^2}\sin\theta+c$$

$$\Rightarrow I=\frac{x}{a^2(x^2+a^2)^{1/2}}+c$$

Example 5 Evaluate $\int \sqrt{\frac{1-x}{1+x}}\,dx.$

Solution:

$$I=\int \sqrt{\frac{1-x}{1+x}}\,dx$$

Put $x=\cos 2\theta$. Then $dx=-2\sin 2\theta\cdot d\theta.$

$$I=-2\int \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\sin 2\theta\cdot d\theta$$

$$\Rightarrow I=-2\int \sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}\sin 2\theta\cdot d\theta=-4\int \tan\theta\cdot\sin\theta\cdot\cos\theta\cdot d\theta$$

$$\Rightarrow I=-4\int \sin^2\theta\, d\theta=-2\int (1-\cos 2\theta)d\theta$$

$$\Rightarrow I=-2\left(\theta-\frac{\sin 2\theta}{2}\right)+c=-2\theta-\sin 2\theta+c$$

$$\Rightarrow I=-\cos^{-1}x+\sqrt{1-x^2}+c$$

- **Prepared by**

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