

ST. LAWRENCE HIGH SCHOOL

A JESUIT CHRISTIAN MINORITY INSTITUTION



STUDY MATERIAL-1 SUBJECT - MATHEMATICS

Pre-test

Chapter: MATRICES AND DETERMINANTS Class: XII

Topic: MATRICES Date: 08.05.2020

PART 1

Matrices

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DEFINITION OF MATRIX

A matrix is any rectangular array of numbers written within brackets. A matrix is usually represented by a capital letter and classified by its dimensions. The dimensions of a matrix are its number of rows and columns. An $m \times n$ matrix is usually written as

$$A_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

where a_{ij} represents any number which lies on the *i*th row (from top) and *j*th column (from left).

Note The matrix is not a number. It has got no numerical value.

TYPES OF MATRICES

Row Matrix

A matrix having only one row is called a row matrix.

Examples:

- (b) [4 5 9]
- (a) $[2\ 3\ 5]$ (c) $[a_1\ a_2\ ...\ a_n]$

Column Matrix

A matrix having only one column is called a column matrix. Examples:

(a)
$$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

(b)
$$\begin{bmatrix} 6 \\ 1 \\ 4 \end{bmatrix}$$

(c)
$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$$

Null Matrix or Zero Matrix

A matrix which has all its elements zero is called a *null matrix*. Examples:

(a)
$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$$

(b)
$$[0]_{1\times 1}$$

(c)
$$\left[m \begin{cases} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ n & & & \\ \end{bmatrix}_{m \times r} \right]_{m \times r}$$

Square Matrix

A matrix in which the number of rows is equal to the number of columns is called a *square matrix*. It is denoted by I_nA (where n is the size of the matrix).

Examples:

(b)
$$\begin{bmatrix} \alpha & \beta & \gamma \\ x & y & z \\ a & b & c \end{bmatrix}_{x \ge 3}$$

(c)
$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n \times n}$$

Note

- $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ are called the elements of the principal diagonal.
- A square matrix is a singular matrix if its determinant is zero, otherwise it is a non-singular matrix.

Diagonal Matrix

A square matrix whose all the non-diagonal elements are zero, i.e. $a_{ij} = 0$, $\forall i \neq j$, is called a diagonal matrix. It can be written as dia $(a_{11}a_{12}...a_{nn})$.

Examples:

(a)
$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

(a)
$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$
 (b)
$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

Scalar Matrix

A square matrix $A = [a_{ij}]_{m \times n}$ is called a *scalar matrix* if

(i)
$$a_{ij} = 0$$
, $\forall i \neq j$ and

(ii)
$$a_{ii} = c$$
 for all i, where $c \neq 0$

i.e. a diagonal matrix in which all the diagonal elements are equal is called a scalar matrix.

Example:
$$\begin{bmatrix} k & 0 & \cdots & 0 \\ 0 & k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & k \end{bmatrix}$$

It can be written as dia(kk...k).

Unit or Identity Matrix

A square matrix each of whose diagonal elements is unity and each of whose non-diagonal elements is equal to zero is called a unit matrix.

Example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note The identity matrices of order n are denoted by I_n .

Transpose of Matrices

If $A = [a_{ii}]_{m \times n}$, then the transpose of A, i.e. A^T or $A' = [a_{ii}]_{n \times m}$.

Example: If
$$A = \begin{bmatrix} 4 & 7 & 5 \\ 3 & 6 & 8 \end{bmatrix}$$
, then $A^{T} = \begin{bmatrix} 4 & 3 \\ 7 & 6 \\ 5 & 8 \end{bmatrix}$.

Properties of transpose of matrices

- 1. $(A^T)^T = A$, i.e. the transpose of the transpose of a matrix is the matrix itself.
- 2. $(A+B)^T = A^T + B^T$, i.e. the transpose of the sum of two matrices is the sum of their transposes.
- 3. $(kA)^T = kA^T (k \text{ is a scalar}).$
- 4. $(AB)^T = B^T A^T$, i.e. the transpose of the product of two matrices is the product in reverse order of their transposes.

Equal Matrices

Two matrices are said to be equal if

- (i) they are of the same order
- (ii) the elements at the corresponding positions of the two matrices are equal.

Illustration 1. If
$$\begin{bmatrix} k+5 & 2m+k \\ n-2 & 4p-6 \end{bmatrix} = \begin{bmatrix} -k-3 & 0 \\ 5 & 3p \end{bmatrix}$$
, then find $Example$: $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 5 & 1 & 7 & 0 \end{bmatrix}$

Solution: : Two matrices are equal,

Symmetric Matrices

They are of two types.

Symmetric matrix

A square matrix $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ is called symmetric if $[a_{ij}] = [a_{ji}]$ for all i, j, that is, if $A^T = A$.

Example:
$$\begin{bmatrix} 2 & 1 & 7 \\ 1 & 3 & 9 \\ 7 & 9 & 5 \end{bmatrix}$$

Skew-symmetric matrix

A square matrix $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ is skew symmetric if $a_{ij} = -a_{ji}$ for all i, j, that is, if $A^T = A$.

Example:
$$\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$$

Illustration 2. Any square matrix A is skew symmetric

$$\Leftrightarrow \quad a_{ij} = -a_{ij} \forall i, j \qquad \qquad \Leftrightarrow \quad (A_{ij}) = -(A^T)_{ij} \forall i, j$$

$$\Leftrightarrow \quad A = -A^T \qquad \Leftrightarrow \quad A^T = -A$$

Note Thus, a square matrix A is a skew-symmetric matrix iff $A^T = -A$.

Upper Triangular Matrix

A square matrix whose all the elements below the principal diagonal are zero, i.e. $a_{ij} = 0$, $\forall i > j$, is called an *upper* triangular matrix.

Examples: (a)
$$\begin{bmatrix} 5 & 4 & 3 \\ 0 & 6 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$
 (b)
$$\begin{bmatrix} 2 & 5 & 6 & 8 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Lower Triangular Matrix

A square matrix whose all the elements above the principal diagonal are zero, i.e. $a_{ij} = 0, \forall i < j$, is known as a *lower* triangular matrix.

Example:
$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 5 & 1 & 7 & 0 \\ 6 & 2 & 3 & 1 \end{bmatrix}$$

Orthogonal Matrix

A square matirx A is called orthogonal if $AA^T = A^TA = I$

Example:
$$A = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$$

Note If |A|, i.e. det(A) = 1, then the matrix is called proper.

Conjugate Matrices

If $A = [a_{ii}]$, then $\overline{A} = [a_{ii}]$.

Example:
$$A = \begin{bmatrix} 2+3i & 1+i \\ 2 & 5i \end{bmatrix} \Rightarrow \overline{A} = \begin{bmatrix} 2-3i & 1-i \\ 2 & -5i \end{bmatrix}$$

Unitary Matrix

If $\overline{A^T}A = I$, then A is called a *unitary matrix*.

Illustration 3. Show that $A = \begin{bmatrix} 1+i & 1+2i \\ 2+i & 1-i \end{bmatrix}$ is a unitary

Solution:
$$A^T = \begin{bmatrix} 1+i & 2+i \\ 1+2i & 1-i \end{bmatrix} \implies \overline{A}^T = \begin{bmatrix} 1-i & 2-i \\ 1-2i & 1+i \end{bmatrix}$$

Now
$$\overline{A^T}A = \begin{bmatrix} 1-i & 2-i \\ 1-2i & 1+i \end{bmatrix} \begin{bmatrix} 1+i & 1+2i \\ 2+i & 1-i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

So A is a unitary matrix.

Hermitian Matrix

If a square matrix is A such that $\overline{A^T} = A$, then A is called a Hermitian matrix.

Illustration 4. If
$$A = \begin{bmatrix} l & x+iy & a+ib \\ x-iy & m & p+iq \\ a-ib & p-iq & n \end{bmatrix}$$

show that it is a Hermitian matrix

Solution:
$$(A^{T}) = \begin{bmatrix} l & x-iy & a-ib \\ x+iy & m & p-iq \\ a+ib & p+iq & n \end{bmatrix}$$

$$\Rightarrow (\overline{A}^{T}) = \begin{bmatrix} l & x+iy & a+ib \\ x-iy & m & p+iq \\ a-ib & p-iq & n \end{bmatrix} = A$$

Hence, A is Hermitian.

Note The diagonal elements of a hermitian matrix are necessarily real.

Skew-Hermitian Matrix

A square matrix $A = [a_{ij}]$ is said to be a skew-Hermitian matrix if $a_{ii} = -\overline{a}_{ii}$, $\forall i, j$, i.e. $A^T = -A$.

Illustration 5. If
$$A = \begin{bmatrix} 2i & -2-3i \\ 2-3i & -i \end{bmatrix}$$

show that it is a skew-Hermitian matrix

Solution:
$$A = \begin{bmatrix} 2i & -2-3i \\ 2-3i & -i \end{bmatrix}$$
$$A^{T} = \begin{bmatrix} 2i & 2-3i \\ -2-3i & -i \end{bmatrix}$$
$$\Rightarrow \overline{A^{T}} = \begin{bmatrix} -2i & 2+3i \\ -2+3i & i \end{bmatrix}$$
$$= -\begin{bmatrix} 2i & -2-3i \\ 2-3i & -i \end{bmatrix} = -A$$

Note

- The diagonal elements of a skew-Hermitian matrix are either purely imaginery or zero.
- Every square matrix (with complex elements) can be uniquely expressed as the sum of Hermitian and skew-

Hermitian matrices, i.e.
$$A = \frac{1}{2} \left(A + \overline{A^T} \right) + \frac{1}{2} \left(A - \overline{A^T} \right)$$
.

Submatrix

A submatrix is obtained by deleting some rows and columns.

Example:
$$A = \begin{bmatrix} 2 & 3 & 5 & 6 \\ 5 & 7 & 1 & 0 \\ 2 & 1 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 5 & 6 \\ 5 & 7 & 1 & 0 \\ 2 & 1 & 1 & 7 \end{bmatrix}$$

$$\Rightarrow \text{ Submatrix of } A = \begin{bmatrix} 5 & 7 & 0 \\ 2 & 1 & 7 \end{bmatrix}.$$

Idempotent Matrix

A square matrix A is called *idempotent* if it satisfies the relation $A^2 = A$.

Illustration 6. Show that the matrix $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is

idempotent.

Solution:
$$A^2 = A \cdot A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \times \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

So, A is idempotent.

Periodic Matrix

A square matrix A is called *periodic* if $A^{k+1} = A$, where k is a positive integer. If k is the least positive integer for which $A^{k+1} = A$, then k is said to be the period of A. For k = 1, we get $A^2 = A$ and we call it to be an idempotent matrix.

Nilpotent Matrix

A square matrix A is called *nilpotent matrix* of order m provided it satisfies the relation $A^k = O$ and $A^{k-1} \neq O$, where k is a positive integer and O is a null matrix and k is the order of the nilpotent matrix A.

Illustration 7. Show that $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is a nilpotent matrix of order 3.

Solution: Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$$A^{2} = A \cdot A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+5+6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

$$A^{3} = A^{2} \cdot A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$\therefore A^3 = 0 \quad \text{i.e. } A^k = 0$$

Hence k = 3. So, A is nilpotent of order 3.

Involutory Matrix

A square matrix A is called *involutory* provided it satisfies the relation $A^2 = I$, where I is the identity matrix.

Illustration 8. Show that the matrix $\begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ is

involutory.

Solution:
$$A^2 = A \cdot A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} \times \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence the given matrix A is involutory.

ADDITION OF MATRICES

If $A[a_{ij}]$ and $B[b_{ij}]$ are two matrices of the same order, then A + B is a matrix, each of whose element is the sum of the corresponding elements of A and B. For example,

$$A+B=\left\lceil a_{ij}+b_{ij}\right\rceil$$

Illustration 9. If
$$A = \begin{bmatrix} a & b & c \\ p & q & r \end{bmatrix}$$
 and $B = \begin{bmatrix} 2 & 4 & 3 \\ 7 & 5 & 1 \end{bmatrix}$, find $A + B$.

Solution: If
$$A = \begin{bmatrix} a & b & c \\ p & q & r \end{bmatrix}$$
 and $B = \begin{bmatrix} 2 & 4 & 3 \\ 7 & 5 & 1 \end{bmatrix}$

$$A + B = \begin{bmatrix} a+2 & b+4 & c+3 \\ p+7 & q+5 & r+1 \end{bmatrix}$$

PROPERTIES OF MATRIX ADDITION

If A, B and C are three matrices of the same order $m \times n$ and $A = \begin{bmatrix} a_{ij} \end{bmatrix}$, $B = \begin{bmatrix} b_{ij} \end{bmatrix}$, $C = \begin{bmatrix} c_{ij} \end{bmatrix}$, then the following properties hold in matrix addition:

- 1. Commutative law i.e. A + B = B + A
- 2. Associative law i.e. A + (B + C) = (A + B) + C
- 3. Distributive law i.e. k(A + B) = kA + kB (where k is a scalar)
- 4. Additive identity i.e. A + O = A = O + A
- 5. Additive inverse i.e. A + B = O = B + AHere B is the additive inverse of the matrix A or the negative of A.
- 6. (i) Left cancellation law, i.e.

$$A + B = A + C \implies B = C$$

(ii) Right cancellation law, i.e.

$$B + A = C + A \implies B = C$$

7. The equation A + X = 0 has a unique solution X = -A, where X has the order $m \times n$.

Illustration 10. If
$$A = \begin{bmatrix} 2 & 3 \\ 5 & 2 \\ 8 & 9 \end{bmatrix}$$
, $B = \begin{bmatrix} 2 & 1 \\ 5 & 2 \\ 8 & 9 \end{bmatrix}$ and $A + B - 2D$

= 0, then find D.

Solution: Given $A + B - 2D = 0 \implies D = \frac{1}{2}(A + B)$

$$D = \frac{1}{2} \begin{bmatrix} 2 & 3 \\ 5 & 2 \\ 8 & 9 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 2 & 1 \\ 5 & 2 \\ 8 & 9 \end{bmatrix}$$

 $D = \frac{1}{2} \begin{bmatrix} 4 & 4 \\ 10 & 4 \\ 16 & 18 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 5 & 2 \\ 8 & 9 \end{bmatrix}$

SUBTRACTION OF MATRICES

If $A[a_{ij}]$ and $B[b_{ij}]$ are two matrices of the same order, then $A - B = [a_{ij} - b_{ij}].$

PROPERTIES OF MATRIX SUBTRACTION

The properties of matrix subtraction are the same as those of addition.

Illustration 11. Solve the following equation for x and y:

$$2x - y = \begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{bmatrix} \text{ and } 2y + x = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$$

Solution: $2x - y = \begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{bmatrix}$

Multiplying both sides by 2, we get

$$4x - 2y = \begin{bmatrix} 6 & -6 & 0 \\ 6 & 6 & 4 \end{bmatrix}$$
 (i)

Given $x + 2y = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$ (ii)

From (i) + (ii), we get

$$5x = \begin{bmatrix} 6 & -6 & 0 \\ 6 & 6 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix} = \begin{bmatrix} 10 & -5 & 5 \\ 5 & 10 & 0 \end{bmatrix}$$
$$x = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

Putting the value of x, in Eq. (ii), we get

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix} + 2y = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$$

$$\Rightarrow 2y = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 4 \\ -2 & 2 & -4 \end{bmatrix}$$

$$\Rightarrow y = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$$

$$\Rightarrow x = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$$

Note Two matrices are said to be conformable for addition iff they are of the same order.

MULTIPLICATION OF MATRICES

For multiplication unlike addition, A and B may have different sizes. However, the number of columns of A must be the number of rows of B. Thus, if A is $m \times n$, then B must be $n \times p$ for some p. As we shall see, the size of AB is $m \times p$.

Let $A = [a_{ij}]$ be an $m \times n$ matrix and let $B = [b_{ij}]$ be an $n \times p$ matrix. The product AB is the $m \times p$ matrix $C = [c_{ij}]$ such that $c_{ij} = a_{i1}b_{1i} + a_{i2}b_{2j} + a_{i3}b_{2j} + \cdots + a_{in}b_{nj}$ for $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, \ldots, p$.

Size of A	Size of B	Size of AB
2×3	3×5	2×5
4×2	2 × 3	4×3
3×1	1 × 3	3×3
1×3	3×1	1 × 1
5 × 3	3×5	5 × 5
5 × 3	5 × 3	AB is not defined

Note The following diagram may help you remember the relationship between sizes of matrices when working with a product *AB*.

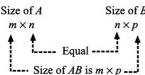


Fig. 4.1

PROPERTIES OF MULTIPLICATION

1. Multiplication of matrices is not commutative, i.e. $AB \neq BA$.

Note If AB = -BA, then A and B are said to be anticommutative.

2. Matrix multiplication is associative if conformability is assured, i.e. A(BC) = (AB)C.

- 3. Matrix multiplication is distributive with respect to addition, i.e. A(B+C) = AB + AC.
- 4. If A and B are two matrices and O is a null matrix such that AB = 0, then A = 0 or B = 0 or both are 0.
- 5. Multiplication of matrix A by unit matrix I.

$$AI = A = IA$$

- 6. If A is an $n \times p$ matrix and O is a null matrix, then
 - (i) $A_{n \times p} O_{p \times m} = O_{n \times m}$
 - (ii) $O_{m \times n} A_{n \times p} = O_{m \times p}$
- 7. The product of $\underbrace{AA...A}_{m \text{ times}} = A^m \text{ and } (A^m)^n = A^{mn}$.

ADJOINT OF A SQUARE MATRIX

Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ be a square matrix of order n and let C_{ij} be a cofactor of a_{ij} in A. Then the transpose of the matrix of cofactors of elements of A is called the adjoint of A which is denoted by adj A. That is, $adj A = \begin{bmatrix} C_{ij} \end{bmatrix}^T \Rightarrow (adj A)_{ij} = C_{ij} = cofactors of <math>a_{ij}$ in A.

Illustration 12. Find the adjoint of matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -3 \\ -1 & 2 & 3 \end{bmatrix}$.

Solution: Let C_{ij} be a cofactor of a_{ij} in A. Then cofactors of elements of A are given by

$$C_{11} = \begin{vmatrix} 1 & -3 \\ 2 & 3 \end{vmatrix} = 9, C_{12} = -\begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix} = -3, C_{13} = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5$$

$$C_{21} = -\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = -1, C_{22} = \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = 4, C_{23} = -\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = -3$$

$$C_{31} = \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = -4, C_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = 5, C_{33} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 9 & -3 & 5 \\ -1 & 4 & -3 \\ -4 & 5 & -1 \end{bmatrix}^T = \begin{bmatrix} 9 & -1 & -4 \\ -3 & 4 & 5 \\ -5 & -3 & -1 \end{bmatrix}$$

PROPERTIES OF adj A

- 1. If A is an *n*-rowed square matrix, then (adj A)A = A(adj A) = $|A|I_n$.
- Case 1. If A is a square singular matrix, then (adj A)A = A(adj A) = O (null matrix).

Case 2. If A is non-singular matrix, then $|adj A| = |A|^{n-1}$.

- 2. $adj(AB) = (adj B) \cdot (adj A)$
- $3. \quad (\operatorname{adj} A)^T = \operatorname{adj} A^T$
- 4. $adj(adjA) = |A|^{n-2} A$ (A is a non-singular matrix)

- 5. $|\operatorname{adj}(\operatorname{adj} A)| = |A|^{(n-1)^2}$ (A is a non-singular matrix)
- 6. Adjoint of a diagonal matrix is a diagonal matrix.

INVERSE OF A MATRIX

Let A be a square matrix of order n. If there exists a matrix B such that $AB = I_n = BA$, then B is called the *inverse* of A and is denoted by A^{-1} (read "A inverse").

TWO THEOREMS RELATED TO INVERTIBLE MATRIX

1. Every invertible matrix possesses a unique inverse. **Proof:** Let A be an invertible matrix of order $n \times m$. Let B and C be two inverse matrices of A

Then
$$AB = BA = I_n$$
 (i)

and
$$AC = CA = I_n$$
 (ii)

Now, $AB = I_n$

$$\Rightarrow C(AB) = CI_n \Rightarrow (CA)B = CI_n$$
(by associativity)

$$\Rightarrow I_n B = CI_n \quad [\because CA = I_n \text{ by (ii)}]$$

$$\Rightarrow B = C \quad [\because I_n B = B, CI_n = C]$$

Hence, an invertible matrix possesses a unique inverse.

2. A square matrix is invertible if and only if it is non-singular.

Proof: Let A be an invertible matrix. Then there exists a matrix B such that

$$AB = I_n = BA \quad \Rightarrow \quad |AB| = |I_n|$$

$$\Rightarrow \quad |A||B| = 1 \quad [\because |AB| = |A||B|]$$

$$\Rightarrow |A| \neq 0 \quad \Rightarrow \quad A \text{ is a non-singular matrix.}$$

A METHOD FOR FINDING THE INVERSE OF A MATRIX

Let A be a non-singular square matrix of order n. Then

$$A(\operatorname{adj} A) = |A|I_n = (\operatorname{adj} A)A$$

$$\Rightarrow A\left(\frac{1}{|A|}\operatorname{adj}A\right) = I_n = \left(\frac{1}{|A|}\operatorname{adj}A\right)A$$

$$\Rightarrow A^{-1} = \frac{1}{|A|} \operatorname{adj} A \quad \text{[by definition of inverse]}$$

MORE RESULTS TO INVERSE OF A MATRIX

If A, B and C are non-singular square matrices, then

1. (i)
$$AB = AC \implies B = C$$

(ii)
$$BA = CA \implies B = C$$

- 2. $(AB)^{-1} = B^{-1}A^{-1}$ in general $(ABC...Z)^{-1} = Z^{-1}Y^{-1}...B^{-1}A^{-1}$
- 3. $(A^T)^{-1} = (A^{-1})^T$
- $4. \left(\overline{A^T}\right)^{-1} = \left(\overline{A^{-1}}\right)^T$
- 5. Inverse of a non-singular diagonal matrix

If
$$A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$$
 (and $|A| \neq 0$), then $A^{-1} = \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & \frac{1}{y} & 0 \\ 0 & 0 & \frac{1}{z} \end{bmatrix}$

FINDING THE INVERSE OF A SQUARE MATRIX BY USING ELEMENTARY ROW TRANSFORMATIONS

For a 2×2 Matrix

Step I. Make $a_{11} = 1$ by operating either $R_1 \to \frac{1}{a_{11}} R_1$ (if $a_{11} \neq 0$) or by $R_1 \leftrightarrow R_2$. In case of 0, then first operate $R_1 \leftrightarrow R_2$ and then

$$R_1 \rightarrow \frac{1}{a_{11}} R_1$$
 (if need be).

- **Step II.** Make $a_{12} = 0$ by the operation $R_1 \rightarrow R_2 a_{21}R_1$.
- **Step III.** Make $a_{22} = 1$ by the operation $R_2 \rightarrow \frac{1}{a_1} R_2$.
- **Step IV.** Make $a_{12} = 0$ by the operation $R_1 \rightarrow R_1 a_{12}R_2$.

For a 3 × 3 Matrix

- **Step I.** Make $a_{11} = 1$ either by operating $R_1 \to \frac{1}{a_{11}}$ (if $a_{11} \neq 0$) or by operating $R_1 \leftrightarrow R_2$ or $R_1 \leftrightarrow R_3$ and then $R_1 \to \frac{1}{a_{11}} R_1$ (if need be).
- **Step II.** Operate $R_2 \rightarrow R_2 a_{21}R_1$ and $R_3 \rightarrow R_3 a_{31}R_1$ to obtain $a_{21} = a_{31} = 0$.
- **Step III.** Operate $R_2 \to \frac{1}{a_{22}} R_2$ (if $a_{22} \neq 0$) or $R_2 \leftrightarrow R_3$ following by $R_2 \to \frac{1}{a_{22}} R_2$ (if need be) to make $a_{22} = 1$.
- **Step IV.** Operate $R_3 \to R_3 a_{32}R_2$ to make $a_{32} = 0$.
- **Step V.** Operate $R_3 \rightarrow \frac{1}{a_1} R_3$ to make $a_{33} = 1$.
- **Step VI.** Operate $R_2 \to R_2 a_{23}R_3$ and $R_1 \to R_1 a_{13}R_3$ to make $a_{23} = a_{13} = 0$.

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Step VII. Operate $R_1 \rightarrow R_1 - a_{12}R_2$ to make $a_{12} = 0$.

SOLUTION OF A SYSTEM OF NON-HOMOGENEOUS EQUATIONS

Consider the following system of m linear equations (non-homogenous) in n unknowns $x_1, x_2, ..., x_n$:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_m = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_m = b_2$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_n$$

This system of equations can be written in a matrix form as follows:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Thus, AX = B. Here, A is a square matrix.

$$X = A^{-1}B$$

System of non-homogeneous linear equations

S. No.	Condition	Sub-condition	Nature of solution	Number of solutions	Solution
1.	$ A \neq 0$		Consistent	Unique solution	$X = A^{-1}B$
2.	A = 0	(i) $(\operatorname{Adj} A)B \neq 0$ (ii) $(\operatorname{Adj} A)B = 0$	Inconsistent Consistent	No solution Infinite solutions	

SOLUTION OF A SYSTEM OF HOMOGENEOUS AND LINEAR EQUATIONS

Consider the following system of homogeneous linear equations in n unknowns $x_1, x_2, ..., x_n$.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0 \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0 \end{cases}$$

This system can be written in a matrix form as follows:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Thus,
$$AX = 0$$

System of homogeneous linear equations

S. No.	Condition	Sub condition	Nature of solution	Solution
1.	$ A \neq 0$		Trivial solution	$X = A^{-1}0$
2.	A = 0	(i) $(Adj A)B = 0$ or $B = 0$	Non-trivial solution Infinite solutions	

TRACE OF A MATRIX

If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, then the *trace* of the matrix A is

represented as Tr(A) = a + e + i =the sum of the main diagonal entries.

RANK OF A MATRIX

The rank of a matrix A is the order of the highest-order non-zero minors of A.

- \therefore Rank of a matrix is r if
 - 1. Every minor of order (r+1) is zero.
- 2. There is at least one non-zero minor of order r.

SOLVED PROBLEMS

1. If
$$\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix} \begin{bmatrix} 1 & \tan\theta \\ -\tan\theta & 1 \end{bmatrix}^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$
, then

- (a) a = 1, b = 1
- (b) $a = \cos 2\theta$, $b = \sin 2\theta$
- (c) $a = \sin 2\theta$, $b = \cos 2\theta$ (d) none of these

Ans. (b)

Solution: Since

$$\begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix} \begin{bmatrix} 1 & \tan\theta \\ -\tan\theta & 1 \end{bmatrix}^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix} \frac{1}{1+\tan^2\theta} \begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$\Rightarrow \frac{1}{1+\tan^2\theta} \begin{bmatrix} 1-\tan^2\theta & -2\tan\theta \\ 2\tan\theta & 1-\tan^2\theta \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{1-\tan^2\theta}{1+\tan^2\theta} & \frac{-2\tan\theta}{1+\tan^2\theta} \\ \frac{2\tan\theta}{1+\tan^2\theta} & \frac{1-\tan^2\theta}{1+\tan^2\theta} \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

- $\therefore \quad a = \cos 2\theta, \ b = \sin 2\theta$
- 2. The inverse of a skew-symmetric matrix of an odd order is
 - (a) a symmetric matrix
- (b) a skew-symmetric matrix
- (c) diagonal matrix
- (d) does not exist

Ans. (d)

Solution: Let A be a skew-symmetric matrix of order n. By definition $A^T = -A$

$$\Rightarrow |A^{T}| = |-A| \Rightarrow |A| = (-1)^{n} |A|$$

$$\Rightarrow |A| = -|A| \quad [\because n \text{ is odd}]$$

$$\Rightarrow 2|A| = 0 \Rightarrow |A| = 0$$

- \therefore A^{-1} does not exist.
- 3. Let A be a non-singular square matrix. Then $|\operatorname{adj} A|$ is equal to
 - (a) $|A|^n$
- (b) $|A|^{n-1}$
- (c) $|A|^{n-2}$
- (d) none of these

Ans. (b)

Solution: Since A is non-singular

$$\therefore$$
 A^{-1} exists

Now,
$$A(\operatorname{adj} A) = |A|I = (\operatorname{adj} A)A$$

$$\Rightarrow$$
 $|A||\operatorname{adj} A| = |A|^n = |\operatorname{adj} A||A|$

$$\therefore$$
 $|adjA| = |A|^{n-1}$

- **4.** If A and B are two non-zero square matrices of the same order such that the product AB = 0, then
 - (a) both A and B must be singular
 - (b) exactly one of them must be singular
 - (c) both of them are singular
 - (d) none of these

Ans. (c)

Solution: If possible assume that A is non-singular, then A^{-1} exists. Thus,

$$AB = 0$$
 \Rightarrow $A^{-1}(AB) = (A^{-1}A)B = 0$

 \Rightarrow IB = 0 or B = 0. A contradiction.

- 5. If x + y + z = 6, x y + z = 2, 2x + y z = 1, then x, y, z are respectively
 - (a) 3, 2, 1
- (b) 1, 2, 3
- (c) 2, 1, 3
- (d) none of these

Ans. (b)

Solution:
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$
Let
$$AX = B \implies X = A^{-1}B$$
 (i)

ow, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$

$$|A| = D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 2R_1$

$$\Rightarrow |A| = D = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & -1 & -3 \end{vmatrix} = 6 \neq 0$$

i.e. matrix A is non-singular.

:. We will have a unique solution and the equations are consistent.

Now,
$$A^{-1} = \operatorname{adj} A \div |A| = \frac{1}{6} \begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{6} & -\frac{1}{3} \end{bmatrix} \quad \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{6} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \cdot 0 + \frac{2}{3} + \frac{1}{3} \\ \frac{6}{2} - 1 + 0 \\ \frac{6}{2} + \frac{1}{3} - \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\therefore$$
 $x = 1, y = 2, z = 3$

6. The value of x for which the matrix

$$A = \begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$
 is the inverse of
$$B = \begin{bmatrix} -x & 14x & 7x \\ 0 & 1 & 0 \\ x & -4x & -2x \end{bmatrix}$$

(a)
$$\frac{1}{2}$$

(b)
$$\frac{1}{3}$$

(c)
$$\frac{1}{4}$$

(d)
$$\frac{1}{5}$$

Ans. (d)

Solution: We have
$$AB = \begin{bmatrix} 5x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10x - 2 & 5x \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow$$
 $x =$

7. Let
$$F(\alpha) = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
, where $\alpha \in \mathbb{R}$. Then

 $[F(\alpha)]^{-1}$ is equal to

(a)
$$F(-\alpha)$$

(b)
$$F(\alpha^{-1})$$

(c)
$$F(2\alpha)$$

(d) none of these

Ans. (a)

Solution: Now,

$$F(\alpha)F(-\alpha) = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$F(\alpha)F(-\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\therefore F(-\alpha) = [F(\alpha)]^{-1}$$

8. If
$$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$
 and $A^2 = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$, then

(a)
$$\alpha = a^2 + b^2, \beta = 2ab$$
 (b) $\alpha = a^2 + b^2, \beta = a^2 - b^2$

(b)
$$\alpha = a^2 + b^2, \beta = a^2 - b^2$$

(c)
$$\alpha = 2ab, \beta = a^2 + b^2$$
 (d) $\alpha = a^2 + b^2, \beta = ab$

(d)
$$\alpha = a^2 + b^2$$
, $\beta = ab$

Ans. (a)

Solution: We have

$$A^{2} = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} a^{2} + b^{2} & 2ab \\ 2ab & b^{2} + a^{2} \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$$

$$\Rightarrow \alpha = a^2 + b^2, \beta = 2ab$$

9. If
$$E(\theta) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$
, then $E(\alpha)E(\beta)$ is equal to

(a)
$$E(\alpha^0)$$

(b)
$$E(\alpha\beta)$$

(c)
$$E(\alpha + \beta)$$

(d)
$$E(\alpha - \beta)$$

Solution: Since
$$E(\theta) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

$$\Rightarrow E(\alpha)E(\beta) = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{bmatrix}$$
$$\Rightarrow E(\alpha)E(\beta) = \begin{bmatrix} \cos(\alpha + \beta) & \sin(\alpha + \beta) \\ -\sin(\alpha + \beta) & \cos(\alpha + \beta) \end{bmatrix}$$

$$\Rightarrow E(\alpha)E(\beta) = \begin{bmatrix} \cos(\alpha + \beta) & \sin(\alpha + \beta) \\ -\sin(\alpha + \beta) & \cos(\alpha + \beta) \end{bmatrix}$$

$$\therefore E(\alpha)E(\beta) = E(\alpha + \beta)$$

10. If
$$A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$$
, then the system of equations

$$a_1x + b_1y + c_1z = 0$$
, $a_2x + b_2y + c_2z = 0$

and $a_3x + b_3y + c_3z = 0$ has

- (a) only one solution
- (b) infinite number of solutions
- (c) no solution
- (d) more than one but finite number of solutions

Ans. (a)

Solution: Note that A is invertible as $|A| \neq 0$.

Now,
$$AX = 0 \implies A^{-1}(AX) = 0$$

$$\Rightarrow$$
 $(A^{-1}A)X = 0$ or $IX = 0$ or $X = 0$

 \therefore x = y = z = 0 is the only solution of the system of equations.

11. If
$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$
, then $A^2 + 3A$ equals

- (a) $18I_3$
- (b) 8A
- (c) 12A
- (d) none of these

Ans. (a)

Solution:
$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 3I_3$$

$$A^2 = 9I_3 = 3A$$

$$A^2 + 3A = 3A + 3A = 6A = 6 \times 3I_3 = 18I_3$$

12. If
$$A = \begin{bmatrix} 1 & 0 \\ (1/2) & 1 \end{bmatrix}$$
, then A^{100} is equal to

(a)
$$\begin{bmatrix} 1 & 0 \\ 50 & 1 \end{bmatrix}$$

(a)
$$\begin{bmatrix} 1 & 0 \\ 50 & 1 \end{bmatrix}$$
 (b) $\begin{bmatrix} 1 & 0 \\ (1/2)^{100} & 1 \end{bmatrix}$

(c)
$$\begin{bmatrix} 1 & 0 \\ 25 & 0 \end{bmatrix}$$

(d) none of these

Ans. (a)

Solution:
$$A^2 = \begin{bmatrix} 1 & 0 \\ (1/2) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (1/2) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2(1/2) & 1 \end{bmatrix}$$

 $A^3 = A^2 A = \begin{bmatrix} 1 & 0 \\ 2(1/2) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (1/2) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3(1/2) & 1 \end{bmatrix}$

Continuing in this way, we

$$A^{100} = \begin{bmatrix} 1 & 0 \\ 100(1/2) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 50 & 1 \end{bmatrix}$$

13. If
$$A = \begin{bmatrix} 1 & -5 & 7 \\ 0 & 7 & 9 \\ 11 & 8 & 9 \end{bmatrix}$$
, then the trace of matrix A is

(c) 3

Solution: We know that if $A = [a_{ij}]$ is a square matrix of order $n \times n$, then trace of A is the sum of all diagonal elements.

$$\Rightarrow \qquad \text{tr}(A) = \sum_{i=j}^{n} a_{ij}$$

$$\therefore \quad \text{If } A = \begin{bmatrix} 1 & -5 & 7 \\ 0 & 7 & 9 \\ 11 & 8 & 9 \end{bmatrix}, \text{ then } \text{tr}(A) = 1 + 7 + 9 = 17.$$

14. If A and B are square matrices of order 3 such that |A| = -1, |B| = 3, then the determinant of 3AB is equal to

$$(a) -9$$

(b)
$$-27$$

$$(c) -81$$

Ans. (c)

Solution: det(3AB) = 27 det(AB)

[: order of
$$A = \text{order of } B = 3$$
]
= $27|A||B| = 27(-1)(3) = -81$

15. If $X = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then the value of X^n is

(a)
$$\begin{bmatrix} 3n & -4n \\ n & -n \end{bmatrix}$$

(a)
$$\begin{bmatrix} 3n & -4n \\ n & -n \end{bmatrix}$$
 (b)
$$\begin{bmatrix} 2+n & 5-n \\ n & -n \end{bmatrix}$$

(c)
$$\begin{bmatrix} 3^n & (-4)^n \\ 1^n & (-1)^n \end{bmatrix}$$
 (d) none of these

Ans. (d)

Solution: For n = 2

$$X^{2} = X \cdot X = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 1 + (2 \times 2) & (-2)^{2+1} \\ 2 & 1 - (2 \times 2) \end{bmatrix}$$

$$X^{3} = X^{2} \cdot X = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 7 & -16 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 1 + (2 \times 3) & (-2)^{3+1} \\ 3 & 1 - (2 \times 3) \end{bmatrix}$$

$$\Rightarrow X^n = \begin{bmatrix} 1 + 2n & (-2)^{n+1} \\ n & 1 - 2n \end{bmatrix}$$

So, option (d) is corre

16.
$$A = \begin{bmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & R \end{bmatrix}_{3\times3}$$
, where P, Q and R are non-zero real

numbers, then A^{-1} equals

(a)
$$\frac{1}{PQR}I_3$$

(b)
$$\frac{1}{POR}$$

(c)
$$\frac{1}{PQR}\begin{vmatrix} 1/P & 0 & 0 \\ 0 & 1/Q & 0 \\ 0 & 0 & 1/R \end{vmatrix}$$
 (d)
$$\begin{vmatrix} 1/P & 0 & 0 \\ 0 & 1/Q & 0 \\ 0 & 0 & 1/R \end{vmatrix}$$

(d)
$$\begin{vmatrix} 1/P & 0 & 0 \\ 0 & 1/Q & 0 \\ 0 & 0 & 1/R \end{vmatrix}$$

Ans. (d)

Solution: For the given matrix A, |A| = PQR and

$$adjA = \begin{bmatrix} QR & 0 & 0 \\ 0 & PR & 0 \\ 0 & 0 & PQ \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} (adj A)$$

$$= \frac{1}{PQR} \begin{bmatrix} QR & 0 & 0 \\ 0 & PR & 0 \\ 0 & 0 & PQ \end{bmatrix} = \begin{bmatrix} 1/P & 0 & 0 \\ 0 & 1/Q & 0 \\ 0 & 0 & 1/P \end{bmatrix}$$

Shortcut method:

In a matrix $A = [a_{ij}]_{m \times m}$ (i.e. A is a square matrix), if $a_{ii} \neq 0, \forall i = j$ and $a_{ii} = 0, \forall i \neq j$, then to get A^{-1} only write the reciprocal entry of a_{ij} at that place where it exists.

$$A^{-1} = \begin{bmatrix} 1/P & 0 & 0 \\ 0 & 1/Q & 0 \\ 0 & 0 & 1/R \end{bmatrix}$$

17. If A and B are two square matrices such that $B = -A^{-1}BA$, then $(A + B)^2$ is equal to

- (a) $A^2 + B^2$
- (b) 0
- (c) $A^2 + 2AB + B^2$
- (d) A + B

Ans. (a)

Solution: As $B = -A^{-1}BA$, we get AB = -BA or AB + BA = 0Now, $(A + B)^2 = (A + B)(A + B) = A^2 + AB + BA + B^2$ $= A^2 + 0 + B^2 = A^2 + B^2$

To be continued ...

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