STUDY MATERIAL-13

SUBJECT – MATHEMATICS

Pre-test

Chapter: Differentiation **Class: XII**

Topic: Differentiation **Date: 18.06.2020**

-: Differentiation (Part II) :-

3. Derivative of implicit functions

For the implicit function f(x,y) = 0, differentiate each term with respect to x treating y as a function of x and then collect the terms of dy/dx together on left hand side and remaining terms on the right hand side and then find dy/dx.

Example 1.

If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ then find $\frac{dy}{dx}$.

Solution

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Differentiating both side with respect to x,

$$2ax + 2h\left(x\frac{dy}{dx} + y\right) + 2by\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

$$(2ax + 2hy + 2g) + (2hx + 2by + 2f)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{(ax + hy + g)}{(hx + by + f)}.$$

Example 2.

If $x^3 + y^3 = 3axy$, then find $\frac{dy}{dx}$.

Solution

$$x^3 + y^3 = 3axy$$

Differentiating both sides with respect to x,

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}(3axy)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left[x\frac{dy}{dx} + y\right]$$

$$x^2 + y^2\frac{dy}{dx} = ax\frac{dy}{dx} + ay$$

$$\left(y^2 - ax\right)\frac{dy}{dx} = (ay - x^2)$$

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

Example 3.

Find $\frac{dy}{dx}$ at (1,1) to the curve $2x^2 + 3xy + 5y^2 = 10$

Solution

$2x^2 + 3xy + 5y^2 = 10$

Differentiating both sides with respect to x,

$$\frac{d}{dx}\left[2x^2 + 3xy + 5y^2\right] = \frac{d}{dx}[10]$$

$$4x + 3x\frac{dy}{dx} + 3y + 10y\frac{dy}{dx} = 0$$

$$(3x+10y)\frac{dy}{dx} = -3y - 4x$$

$$\frac{dy}{dx} = -\frac{(3y+4x)}{(3x+10y)}$$

Now, $\frac{dy}{dx}$ at (1, 1) $= -\frac{3+4}{3+10}$

$$= -\frac{7}{13}$$

Example 4.

If $\sin y = x \sin(a+y)$, then prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$.

Solution

$$\sin y = x \sin(a+y)$$

$$x = \frac{\sin y}{\sin(a+y)}$$

Differentiating with respect to *y*,

$$\frac{dx}{dy} = \frac{\sin(a+y)\cos y - \sin y \cdot \cos(a+y)}{\sin^2(a+y)}$$

$$= \frac{\sin(a+y-y)}{\sin^2(a+y)}$$

$$= \frac{\sin a}{\sin^2(a+y)}$$

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

4. Logarithmic differentiation

Sometimes, the function whose derivative is required involves products, quotients, and powers. For such cases, differentiation can be carried out more conveniently if we take logarithms and simplify before differentiation.

Example 5.

Differentiate the following with respect to x.

(i) x^x (ii) $(\log x)^{\cos x}$

Solution

(i) Let $y = x^x$

Taking logarithm on both sides

$$\log y = x \log x$$

Differentiating with respect to x,

$$\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1$$

$$\frac{dy}{dx} = y[1 + \log x]$$

$$\frac{dy}{dx} = x^x[1 + \log x]$$

(ii) Let $y = (\log x)^{\cos x}$

Taking logarithm on both sides

$$\therefore \quad \log y = \cos x \ \log(\log x)$$

Differentiating with respect to x,

$$\frac{1}{y} \cdot \frac{dy}{dx} = \cos x \frac{1}{\log x} \cdot \frac{1}{x} + [\log(\log x)](-\sin x)$$

$$\frac{dy}{dx} = y\left[\frac{\cos x}{x \log x} - \sin x \ \log(\log x)\right]$$

$$= (\log x)^{\cos x}\left[\frac{\cos x}{x \log x} - \sin x \log(\log x)\right]$$

Example 6.

If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$

Solution

$$x^y = e^{x-y}$$

Taking logarithm on both sides,

$$y \log x = (x-y)$$

$$y(1+\log x) = x$$

$$y = \frac{x}{1+\log x}$$

Differentiating with respect to x

$$\frac{dy}{dx} = \frac{(1+\log x)(1) - x\left(0+\frac{1}{x}\right)}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}$$

Example 7.

Differentiate: $\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}$

Solution

Let $y = \sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}} = \left[\frac{(x-3)(x^2+4)}{3x^2+4x+5}\right]^{\frac{1}{2}}$

Taking logarithm on both sides,

$$\log y = \frac{1}{2}\left[\log(x-3)+\log(x^2+4)-\log(3x^2+4x+5)\right]$$

$$\left[\because \log ab = \log a+\log b \quad \text{and} \quad \log\frac{a}{b}=\log a-\log b\right]$$

Differentiating with respect to x,

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x-3}+\frac{2x}{x^2+4}-\frac{6x+4}{3x^2+4x+5}\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}\left[\frac{1}{x-3}+\frac{2x}{x^2+4}-\frac{6x+4}{3x^2+4x+5}\right]$$

5. Differentiation of parametric functions

If the variables x and y are functions of another variable namely t, then the functions are called a parametric functions. The variable t is called the parameter of the function.

If $x = f(t)$ and $y = g(t)$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

6. Differentiation of a function with respect to another function

Let $u = f(x)$ and $v = g(x)$ be two functions of x. The derivative of $f(x)$ with respect to $g(x)$ is given by the formula,

$$\frac{d(f(x))}{d(g(x))} = \frac{du/dx}{dv/dx}$$

Example 8.

Find $\frac{dy}{dx}$ if (i) $x = at^2$, $y = 2at$ (ii) $x = a\cos\theta$, $y = a\sin\theta$

Solution

(i) $x = at^2$, $\frac{dx}{dt} = 2at$ | $y = 2at$, $\frac{dy}{dt} = 2a$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

(ii) $x = a\cos\theta$, $y = a\sin\theta$

$\frac{dx}{d\theta} = -a\sin\theta$ | $\frac{dy}{d\theta} = a\cos\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\cos\theta}{-a\sin\theta} = -\cot\theta$$

Example 9.

Differentiate $\frac{x^2}{1+x^2}$ with respect to x^2

Solution

Let $u = \frac{x^2}{1+x^2}$

$$\frac{du}{dx} = \frac{(1+x^2)(2x) - x^2(2x)}{(1+x^2)^2}$$

$$= \frac{2x}{(1+x^2)^2}$$

$v = x^2$

$$\frac{dv}{dx} = 2x$$

$$\frac{du}{dv} = \frac{\left(\frac{du}{dx}\right)}{\left(\frac{dv}{dx}\right)}$$

$$= \frac{\left[\frac{2x}{(1+x^2)^2}\right]}{2x}$$

$$= \frac{1}{(1+x^2)^2}$$

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