ST. LAWRENCE HIGH SCHOOL

A JESUIT CHRISTIAN MINORITY INSTITUTION

FOR GOD AND COUNTRY

STUDY MATERIAL-2

SUBJECT – MATHEMATICS

1st term

Chapter: Trigonometry **Class: XI**

Topic: Compound angles **Date: 18.06.2020**

1. Compound angles

When we add or subtract angles, the result is called a compound angle. i.e.,the algebraic sum of two or more angles are called compound angles

For example, If A, B, C are three angles then $A \pm B$, $A + B + C$, $A - B + C$ etc., are compound angles.

2. Sum and difference formulae of sine, cosine and tangent

(i) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

(ii) $\sin(A-B) = \sin A \cos B - \cos A \sin B$

(iii) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

(iv) $\cos(A-B) = \cos A \cos B + \sin A \sin B$

(v) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(vi) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(vii) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

(viii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

(ix) $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot B + \cot A}$

(X) $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$

(xi) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

Example 1.

If $\cos A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$, $\frac{3\pi}{2} < A, B < 2\pi$, find the value of

(i) $\sin(A-B)$ (ii) $\cos(A+B)$.

Solution

Since $\frac{3\pi}{2} < A, B < 2\pi$, both A and B lie in the fourth quadrant,

$\therefore \sin A$ and $\sin B$ are negative.

Given $\cos A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$,

Therefore, $\sin A = -\sqrt{1-\cos^2 A}$

$$= -\sqrt{1-\frac{16}{25}}$$
$$= -\sqrt{\frac{25-16}{25}}$$
$$= -\frac{3}{5}$$

Sin B $= -\sqrt{1-\cos^2 B}$

$$= -\sqrt{1-\frac{144}{169}}$$
$$= -\sqrt{\frac{169-144}{169}}$$
$$= -\frac{5}{13}$$

(i) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$= \frac{4}{5}\times\frac{12}{13} - \left(\frac{-3}{5}\right)\times\left(\frac{-5}{13}\right)$$
$$= \frac{48}{65} - \frac{15}{65} = \frac{33}{65}$$

(ii) $\sin(A-B) = \sin A \cos B - \cos A \sin B$

$$= \left(\frac{-3}{5}\right)\left(\frac{12}{13}\right) - \left(\frac{4}{5}\right)\left(\frac{-5}{13}\right)$$
$$= \frac{-36}{65} + \frac{20}{65} = \frac{-16}{65}$$

Example 2.

Find the values of each of the following trigonometric ratios.

(i) $\sin 15^\circ$ (ii) $\cos(-105^\circ)$ (iii) $\tan 75^\circ$ (iv) $-\sec 165^\circ$

Solution

(i) $\sin 15^\circ = \sin(45^\circ - 30^\circ)$

$$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

(ii) $\cos(-105^\circ) = \cos 105^\circ$

$$= \cos(60^\circ + 45^\circ)$$

$$= \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ$$

$$= \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} = \frac{1-\sqrt{3}}{2\sqrt{2}}$$

(iii) $\tan 75^\circ = \tan(45^\circ + 30^\circ)$

$$= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}$$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2 + \sqrt{3}$$

(iv) $\cos 165^\circ = \cos(180^\circ - 15^\circ)$

$$= -\cos 15^\circ$$

$$= -\cos(60^\circ - 45^\circ)$$

$$= -(\cos 60^\circ \cos 45^\circ + \sin 60^\circ \sin 45^\circ)$$

$$= -\left(\frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}\right)$$

$$= -\left(\frac{1+\sqrt{3}}{2\sqrt{2}}\right)$$

$$\therefore \quad -\sec 165^\circ = \frac{2\sqrt{2}}{1+\sqrt{3}}$$

$$= \frac{2\sqrt{2}}{1+\sqrt{3}} \times \frac{1-\sqrt{3}}{1-\sqrt{3}}$$

$$= \sqrt{2}(\sqrt{3}-1)$$

Example 3.

If $\tan A = m \tan B$, prove that $\dfrac{\sin(A+B)}{\sin(A-B)} = \dfrac{m+1}{m-1}$

Solution

Given $\tan A = m \tan B$

$$\frac{\sin A}{\cos A} = m \frac{\sin B}{\cos B}$$

$$\frac{\sin A \cos B}{\cos A \sin B} = m$$

Applying componendo and dividendo rule, we get

$$\frac{\sin A \cos B + \cos A \sin B}{\sin A \cos B - \cos A \sin B} = \frac{m+1}{m-1}$$

$$\frac{\sin(A+B)}{\sin(A-B)} = \frac{m+1}{m-1},$$ which completes the proof.

Ex. 4 An angle θ is divided into two parts so that the ratio of the tangents of the parts is k; if the difference between the parts be ϕ, prove that, sin ϕ = (k - 1)/(k + 1) sin θ .

Solution:

Let, α and β be the two parts of the angle θ.

Therefore, θ = α + β.

By question, θ = α - β. (assuming α >β)

and tan α/tan β = k

⇒ sin α cos β/sin β cos α = k/1

⇒ (sin α cos β + cos α sin β)/(sin α cos β - cos α sin β) = (k + 1)/(k - 1), [by componendo and dividendo]

⇒ sin (α + β)/sin (α - β) = (k + 1)/(k - 1)

⇒ (k + 1) sin Ø = (k - 1) sin θ, [Since we know that α + β = θ; α + β = ϕ]

⇒ sin ϕ = (k - 1)/(k + 1) sin θ. *Proved.*

Ex. 5 If x + y = z and tan x = k tan y, then prove that sin (x - y) = [(k - 1)/(k + 1)] sin z

Solution:

Given tan x = k tan y

⇒ sin x/cos x = k · sin y/cos y

⇒ sin x cos y/cos x sin y = k/1

Applying componendo and dividend, we get

sin x cos y + cos x sin y/ sin x cos y - cos x sin y = k + 1/k - 1

⇒ sin (x + y)/sin (x – y) = k + 1/k - 1

⇒ sin z/sin (x – y) = k + 1/k - 1, [Since x + y = z given]

⇒ sin (x – y) = [k + 1/k – 1] sin z *Proved.*

Ex. 6 If $A + B + C = \pi$ and $\cos A = \cos B \cos C$, show that, $\tan B \tan C = 2$

Solution:

$A + B + C = \pi$

Therefore, $B + C = \pi - A$

$\Rightarrow \cos (B + C) = \cos (\pi - A)$

$\Rightarrow \cos B \cos C - \sin B \sin C = - \cos A$

$\Rightarrow \cos B \cos C + \cos B \cos C = \sin B \sin C$,[Since we know, $\cos A = \cos B \cos C$]

$\Rightarrow 2 \cos B \cos C = \sin B \sin C$

$\Rightarrow \tan B \tan C = 2$ *Proved.*

Ex. 7 If $\sin (A + B) + \sin (B + C) + \cos (C - A) = -3/2$ show that,

$\sin A + \cos B + \sin C = 0$; $\cos A + \sin B + \cos C = 0$.

Solution:

Since, $\sin (A + B) + \sin (B + C) + \cos (C - A) = -3/2$

Therefore, $2 (\sin A \cos B + \cos A \sin B + \sin B \cos C + \cos B \sin C + \cos C \cos A + \sin C \sin A) = -3$

$\Rightarrow 2 (\sin A \cos B + \cos A \sin B + \sin B \cos C + \cos B \sin C + \cos C \cos A + \sin C \sin A) = - (1 + 1 + 1)$

$\Rightarrow 2 (\sin A \cos B + \cos A \sin B + \sin B \cos C + \cos B \sin C + \cos C \cos A + \sin C \sin A) = - [(\sin^2 A + \cos^2 A) + (\sin^2 B + \cos^2 B) + (\sin^2 C + \cos^2 C)]$

$\Rightarrow (\sin^2 A + \cos^2 B + \sin^2 C + 2 \sin A \sin C + 2 \sin A \cos B + 2 \cos B \sin C) + (\cos^2 A + \sin^2 B + \cos^2 C + 2 \cos A \sin B + 2 \sin B \cos C + 2 \cos A \cos C) = 0$

$\Rightarrow (\sin A + \sin B + \sin C)^2 + (\cos A + \sin B + \cos C)^2$

Now the sum of squares of two real quantities is zero if each quantity is separately zero.

Therefore, $\sin A + \cos B + \sin C = 0$

and $\cos A + \sin B + \cos C = 0$. *Proved.*

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