ST. LAWRENCE HIGH SCHOOL

A JESUIT CHRISTIAN MINORITY INSTITUTION

FOR GOD AND COUNTRY

STUDY MATERIAL-18

SUBJECT – MATHEMATICS

1st - Term

Chapter: Algebra **Class: XI**

Topic: Complex numbers (Part 3) **Date: 03.08.2020**

➢ Conjugate of a Complex Number :-

The complex numbers $z = (a, b) = a + ib$ and $\bar{z} = (a, -b) = a - ib$, where a and b are the real numbers, $i = \sqrt{-1}$ and $b \neq 0$, are called to be complex conjugate of each other. (Here, the complex conjugate is obtained by just changing the sign of i).

Note that,

$$\text{sum} = (a + ib) + (a - ib) = 2a, \text{ which is real}$$

and

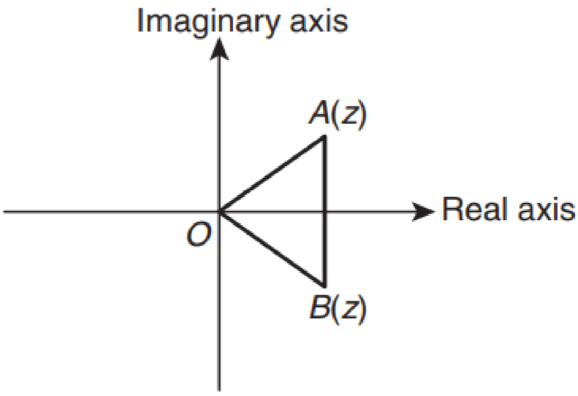
$$\text{product} = (a + ib)(a - ib) = a^2 - (ib)^2 = a^2 - i^2b^2$$
$$= a^2 - (-1)\,b^2 = a^2 + b^2, \text{ which is real}$$

Properties of conjugate

- $\overline{(\bar{z})} = z$
- $z = \bar{z} \Leftrightarrow z$ is real
- $z = -\bar{z} \Leftrightarrow z$ is purely imaginary
- $\text{Re}(z) = \text{Re}(\bar{z}) = \dfrac{z + \bar{z}}{2}$
- $\text{Im}(z) = \dfrac{z - \bar{z}}{2i}$
- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
- $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$
- $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2} \quad (z_2 \neq 0)$
- $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2\text{Re}(\bar{z}_1 z_2) = 2\text{Re}(z_1\bar{z}_2)$
- $\overline{z^n} = (\bar{z})^n$
- If $z = f(z_1)$, then $\bar{z} = f(\bar{z}_1)$

➢ Modulus of a Complex Number :-

Modulus of a complex number $z = x + iy$ is a real number given by $|z| = \sqrt{x^2 + y^2}$. It is always non-negative and $|z| = 0$ only for $z = 0$, that is, origin of the Argand plane. Geometrically, it represents the distance of the point $z(x, y)$ from origin.



Properties of modulus

- $|z| \geq 0 \Rightarrow |z| = 0$ iff $z = 0$, and $|z| > 0$ iff $z \neq 0$.
- $-|z| \leq \text{Re}(z) \leq |z|$, and $-|z| \leq \text{Im}(z) \leq |z|$.
- $|z| = |\bar{z}| = |-z| = |-\bar{z}|$
- $z\bar{z} = |z|^2$
- $|z_1 z_2| = |z_1|\,|z_2|$

  In general, $|z_1 z_2 z_3 \ldots z_n| = |z_1|\,|z_2|\,|z_3| \ldots |z_n|$
- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|} \quad (z_2 \neq 0)$
- $|z_1 \pm z_2| \leq |z_1| + |z_2|$

  In particular, if $|z_1 + z_2| = |z_1| + |z_2|$, then origin, z_1 and z_2 are col-linear with origin at one of the ends.
- $|z_1 \pm z_2| \geq ||z_1| - |z_2||$

  In particular, if $|z_1 - z_2| = ||z_1| - |z_2||$, then origin, z_1 and z_2 are collinear with origin at one of the ends.
- $|z^n| = |z|^n$
- $||z_1| - |z_2|| \leq |z_1| + |z_2|$

  Thus, $|z_1| + |z_2|$ is the greatest possible value of $|z_1 + z_2|$ and $||z_1| - |z_2||$ is the least possible value of $|z_1 + z_2|$.

- $|z_1 \pm z_2|^2 = (z_1 \pm z_2)(\overline{z_1} \pm \overline{z_2}) = |z_1|^2 + |z_2|^2 \pm (z_1\overline{z_2} + \overline{z_1}z_2)$ or $|z_1|^2 + |z_2|^2 \pm 2\text{Re}(z_1\overline{z_2})$
- $z_1\overline{z_2} + \overline{z_1}z_2 = 2|z_1||z_2|\cos(\theta_1 - \theta_2)$ where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$
- $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \dfrac{z_1}{z_2}$ is purely imaginary
- $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}$
- $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$ where $a, b \in R$

➢ Argument of a Complex Number :-

If $z = x + iy = r(\cos\theta + i\sin\theta)$, where $r = \sqrt{x^2 + y^2}$, then θ is called the argument of Z or the amplitude of Z. Since $x = r\cos\theta$ and $y = r\sin\theta$, θ is such that $\cos\theta = \dfrac{x}{\sqrt{x^2 + y^2}}$ and $\sin\theta = \dfrac{y}{\sqrt{x^2 + y^2}}$. Since there can be many values of θ satisfying these conditions, by convention, θ such that $-\pi < \theta \le \pi$ is defined as the principal argument of Z and is denoted by arg Z. The argument of a complex number $a + ib$ is given by $\alpha, \pi - \alpha, -\pi + \alpha$ or $-\alpha$ if $a + ib$ is in the first, second, third or fourth quadrant, respectively, where $\alpha = \tan^{-1}\left|\dfrac{b}{a}\right|$. For example,

- $Z = 1 + i = (1, 1)$ and is marked by point $P(1, 1)$ that lies in first quadrant. Therefore,

  $|Z| = \sqrt{2}$ and $\arg Z = \pi/4$
- If $Z = 1 - i = (1, -1)$, then P lies in the fourth quadrant and $|Z| = \sqrt{2}$ and $\arg Z = -\pi/4$.
- If $Z = -1 + i = (-1, 1)$, then P lies in the second quadrant and $\arg Z = \dfrac{3\pi}{4}$.
- If $Z = -1 - i$, then P lies in the third quadrant and $\arg Z = -\dfrac{3\pi}{4}$.

- Argument of all positive real numbers such as 1, 2, 3, $\frac{1}{2}$, … is 0 since they are marked on the positive x-axis. The argument of all negative real numbers such as −1, −2, −3, … is π since they are marked on negative x-axis. The argument of purely imaginary numbers such as $i, 2i, 3i, \ldots$ is $\frac{\pi}{2}$ since these are marked on the positive y-axis. The argument of purely imaginary numbers like $-i, -2i, -3i, \ldots$ is $-\frac{\pi}{2}$. Since these are marked on negative y-axis.

Properties of arguments

- $\arg(z_1z_2) = \arg(z_1) + \arg(z_2) + 2k\pi\,(k = 0 \text{ or } 1 \text{ or } -1)$

  In general $\arg(z_1z_2z_3 \ldots z_n) = \arg(z_1) + \arg(z_2) + \arg(z_3) + \cdots + \arg(z_n) + 2k\pi$

  (where $k \in I$)
- $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 + 2k\pi$ $\quad (k = 0 \text{ or } 1 \text{ or } -1)$
- $\arg\left(\frac{z}{\bar{z}}\right) = 2\arg z + 2k\pi$ $\quad (k = 0 \text{ or } 1 \text{ or } -1)$
- $\arg(z^n) = n\arg z + 2k\pi$ $\quad (k = 0 \text{ or } 1 \text{ or } -1)$
- If $\arg\left(\frac{z_2}{z_1}\right) = \theta$, then $\arg\left(\frac{z_1}{z_2}\right) = 2k\pi - \theta$ where $k \in I$.
- $\arg \bar{z} = -\arg z$
- If $\arg z = 0$, then z is real.

Note: Proper value of k must be chosen in above results so that arguments lies in $(-\pi, \pi]$.

All the above formulae are written on the basis of the principal argument.

➢ Cube roots of unity :-

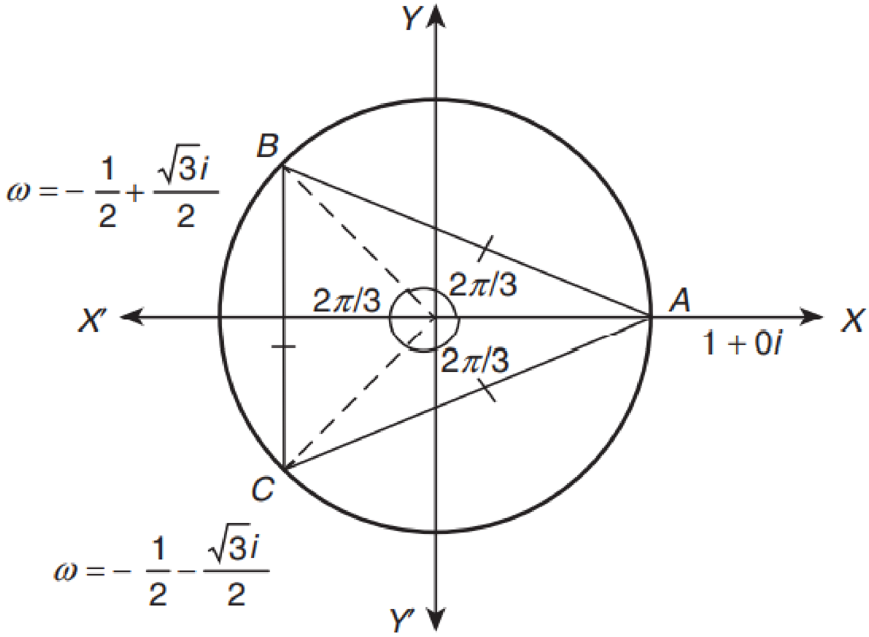
Consider the cubic (third degree) equation

$$x^3 = 1 = \cos 0 + i\sin 0 = \cos 2k\pi + i\sin 2k\pi$$

Therefore,

$$x = \sqrt[3]{1} = (\cos 2k\pi + i\sin 2k\pi)^{1/3}$$

$$= \cos\left(\frac{2k\pi}{3}\right) + i\sin\left(\frac{2k\pi}{3}\right)$$



To get three roots of the cubic equation, we give

$k = 0$, giving the real root, $\cos 0 + i \sin 0 = 1$

$k = 1$, giving one imaginary root, $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \omega$

$k = 2$, giving the other imaginary root, $\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \omega^2$

It is said that $1, \omega, \omega^2$ are the three cubic roots of unity satisfying

(a) $1 + \omega + \omega^2 = 0$

(b) $\omega^3 = 1$

(c) $1, \omega, \omega^2$ are represented respectively by points A, B, C lying on the unit circle $|Z| = 1$ and forming the corners of an equilateral triangle with each side of length $\sqrt{3}$.

Some useful results

$(x^3 + y^3) = (x + y)(x + \omega y)(x + \omega^2 y)$

$(x^3 - y^3) = (x - y)(x - \omega y)(x - \omega^2 y)$

$(x^3 + y^3 + z^3 - 3xyz) = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$

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