ST. LAWRENCE HIGH SCHOOL

A JESUIT CHRISTIAN MINORITY INSTITUTION

STUDY MATERIAL-5

SUBJECT – MATHEMATICS

1st term

Chapter: Trigonometry **Class: XI**

Topic: Trigonometric Identities **Date: 23.06.2020**

Trigonometric Ratios and Identities

(Solved examples) :-

Additional Solved Examples

1. $\sqrt{3}$ cosec 20° − sec 20° = ____.

(A) 1 **(B)** 2

(C) 3 **(D)** 4

Solution:

$$\frac{4\cdot\left(\frac{\sqrt{3}}{2}\cos 20^\circ-\frac{1}{2}\sin 20^\circ\right)}{2\sin 20^\circ\cos 20^\circ}=4\cdot\frac{\sin 60^\circ\cos 20^\circ-\cos 60^\circ\sin 20^\circ}{\sin 40^\circ}$$

$$=4$$

Hence, the correct answer is option (D).

2. $\tan A+2\tan 2A+4\tan 4A+8\cot 8A=$ ____.

(A) $\cot A$ **(B)** $\tan 6A$

(C) $\cot 4A$ **(D)** None of these

Solution:

$$\tan A+2\tan 2A+4\tan 4A+8\cot 8A$$

$$=\tan A+2\tan 2A+4\tan 4A+8\,\frac{1-\tan^2 4A}{2\tan 4A}$$

$$=\tan A+2\tan 2A+4\cot 4A=\tan A+2\tan 2A+4\,\frac{1-\tan^2 2A}{2\tan 2A}$$

$$=\tan A+2\cot 2A=\tan A+\frac{1-\tan^2 A}{\tan A}=\cot \text{A}$$

Hence, the correct answer is option (A).

3. The value of $\sin 12^\circ\cdot\sin 48^\circ\cdot\sin 54^\circ=$ ____.

(A) 1/8 **(B)** 1/6

(C) 1/4 **(D)** 1/2

Solution:

$$\sin 12°\cdot\sin 48°\cdot\sin 54°$$
$$=\frac{1}{2}[\cos 36°-\cos 60°]\sin 54°=\frac{1}{2}\left[\cos 36°\sin 54°-\frac{1}{2}\sin 54°\right]$$
$$=\frac{1}{4}[2\cos 36°\sin 54°-\sin 54°]=\frac{1}{4}[\sin 90°+\sin 18°-\sin 54°]$$
$$=\frac{1}{4}[1-(\sin 54°-\sin 18°)]=\frac{1}{4}[1-2\sin 18°\cos 36°]$$
$$=\frac{1}{4}\left[1-2\frac{\sin 18°\cos 18°\cos 36°}{\cos 18°}\right]=\frac{1}{4}\left[1-\frac{\sin 36°\cos 36°}{\cos 18°}\right]$$
$$=\frac{1}{4}\left[1-\frac{2\sin 36°\cos 36°}{2\cos 18°}\right]=\frac{1}{4}\left[1-\frac{\sin 72°}{2\sin 72°}\right]=\frac{1}{4}\left[1-\frac{1}{2}\right]=\frac{1}{8}$$

Alternative Method

Let $\theta = 12°$

$$\sin 12°\cdot\sin 48°\cdot\sin 54°=\frac{\sin 12°\sin 48°\sin 72°\sin 54°}{\sin 72°}$$
$$=\frac{\sin(3\cdot 12)°\sin 54°}{4\sin 72°}=\frac{\sin(36)°\sin 54°}{8\sin(36)°\cos(36)°}=\frac{\cos 36°}{8\cos(36)°}=\frac{1}{8}=\frac{1}{8}$$

Hence, the correct answer is option (A).

4. If $\sin\theta = 3\sin(\theta+2\alpha)$, then the value of $\tan(\theta+\alpha)+2\tan\alpha$ is

(A) 0 **(B)** 2

(C) 4 **(D)** 1

Solution: Given $\sin\theta = 3\sin(\theta+2\alpha)$. Now

$$\sin(\theta+\alpha-\alpha)=3\sin(\theta+\alpha+\alpha)$$
$$\Rightarrow \sin(\theta+\alpha)\cos\alpha-\cos(\theta+\alpha)\sin\alpha$$
$$=3\sin(\theta+\alpha)\cos\alpha+3\cos(\theta+\alpha)\sin\alpha$$
$$\Rightarrow -2\sin(\theta+\alpha)\cos\alpha=4\cos(\theta+\alpha)\sin\alpha$$
$$\Rightarrow \frac{-\sin(\theta+\alpha)}{\cos(\theta+\alpha)}=\frac{2\sin\alpha}{\cos\alpha}$$
$$\Rightarrow \tan(\theta+\alpha)+2\tan\alpha=0$$

Hence, the correct answer is option (A).

5. The minimum value of $3\tan^2\theta + 12\cot^2\theta$ is

(A) 6 **(B)** 8

(C) 10 **(D)** None of these

Solution:

$$\text{A.M.} \geq \text{G.M} \Rightarrow \frac{1}{2}(3\tan^2\theta+12\cot^2\theta)\geq 6$$
$\Rightarrow 3\tan^2\theta+12\cot^2\theta$ has minimum value 12.

Hence, the correct answer is option (D).

6. Prove that $3(\sin x-\cos x)^4+4(\sin^6 x-\cos^6 x)+6(\sin x+\cos x)^2=13$.

Solution: Let t_1, t_2, t_3 denote the three expressions on the left.

$$t_1=3[(\sin x-\cos x)^2]^2=3(\sin^2 x+\cos^2 x-2\sin x\cos x)^2$$
$$=3(1-2\sin x\cos x)^2=3(1+4\sin^2 x\cos^2 x-4\sin x\cos x)$$
$$t_2=4(\sin^6 x-\cos^6 x)=4(\sin^2 x+\cos^2 x)(\sin^4 x+\cos^4 x-\sin^2 x\cos^2 x)$$
$$=4[(\sin^2 x+\cos^2 x)^2-2\cos^2 x\sin^2 x-\sin^2 x\cos^2 x]$$
$$=4(1-3\sin^2 x\cos^2 x)$$
$$t_3=6(\sin^2 x+\cos^2 x+2\sin x\cos x)=6(1+2\sin x\cos x)$$

Therefore, $t_1+t_2+t_3=3+4+6=13$

7. Prove that $\sin^2\frac{\pi}{8}+\sin^2\frac{3\pi}{8}+\sin^2\frac{5\pi}{8}+\sin^2\frac{7\pi}{8}=2$.

Solution:

$$\sin\frac{7\pi}{8}=\sin\left(\pi-\frac{\pi}{8}\right)=\sin\frac{\pi}{8}$$
$$\sin\frac{5\pi}{8}=\sin\left(\pi-\frac{3\pi}{8}\right)=\sin\frac{3\pi}{8}$$
$$\sin\frac{3\pi}{8}=\sin\left(\frac{\pi}{2}-\frac{\pi}{8}\right)=\cos\frac{\pi}{8}$$

So we have

$$\sin^2\frac{\pi}{8}+\sin^2\frac{3\pi}{8}+\sin^2\frac{5\pi}{8}+\sin^2\frac{7\pi}{8}=2\left(\sin^2\frac{\pi}{8}+\sin^2\frac{3\pi}{8}\right)$$
$$=2\left(\sin^2\frac{\pi}{8}+\cos^2\frac{\pi}{8}\right)=2$$

8. Prove the identity:

$(\cos A+\cos B)(\cos 2A+\cos 2B)(\cos 2^2A+\cos 2^2B)\cdots(\cos 2^{n-1}A+\cos 2^{n-1}B)=\dfrac{(\cos 2^nA-\cos 2^nB)}{2^n(\cos A-\cos B)}$.

Solution:

$$(\cos A-\cos B)(\cos A+\cos B)=\cos^2 A-\cos^2 B=\frac{1}{2}[(1+\cos 2A)-(1+\cos 2B)]$$
$$=\frac{1}{2}(\cos 2A-\cos 2B) \qquad (1)$$

Therefore,

$$(\cos A-\cos B)(\cos A+\cos B)(\cos 2A+\cos 2B)$$
$$=\frac{1}{2}[(\cos 2A-\cos 2B)(\cos 2A+\cos 2B)]=\frac{1}{2^2}(\cos 2^2A-\cos 2^2B)$$

Therefore,

$$(\cos A-\cos B)(\cos A+\cos B)(\cos 2A+\cos 2B)(\cos 2^2A+\cos 2^2B)$$

Proceeding in this manner, we get

$$(\cos A-\cos B)(\cos A+\cos B)(\cos 2A+\cos 2B)(\cos 2^2A+\cos 2^2B)\cdots(\cos 2^{n-1}A+\cos 2^{n-1}B)$$
$$=\frac{1}{2^n}(\cos 2^nA-\cos 2^nB)$$

Hence, the given identity follows.

9. Show that $\dfrac{\sin 8A}{2\sin A}=\cos A+\cos 3A+\cos 5A+\cos 7A$

Solution:

$$\text{R.H.S.}=(\cos A+\cos 3A)+(\cos 5A+\cos 7A)$$
$$=[\cos(2A-A)+\cos(2A+A)]+[\cos(6A-A)+\cos(6A+A)]$$
$$=2\cos A\cdot\cos 2A+2\cos A\cdot\cos 6A$$
$$=2\cos A[\cos(4A-2A)+\cos(4A+2A)]=2\cos A\cdot 2\cos 2A\cdot\cos 4A$$
$$=\frac{(2\sin A\cos A)}{\sin A}\cdot 2\cos 2A\cdot\cos 4A$$
$$=\frac{(2\sin 2A\cdot\cos 2A)\cdot\cos 4A}{\sin A}=\frac{\sin 4A\cdot\cos 4A}{\sin A}=\frac{\sin 8A}{2\sin A}$$

10. Prove that $\cos^2 x+\cos^2\left(\frac{\pi}{3}+x\right)-\cos x\cos\left(\frac{\pi}{3}+x\right)$ is independent of x.

Solution:

$$\cos^2 x+\cos^2\left(\frac{\pi}{3}+x\right)-\cos\left(\frac{\pi}{3}+x\right)\cos x$$

$$=\frac{1}{2}\left(2\cos^2 x+2\cos^2\left(\frac{\pi}{3}+x\right)-2\cos\left(\frac{\pi}{3}+x\right)\cos x\right)$$

$$=\frac{1}{2}\left(1+\cos\left(\frac{2\pi}{3}+2x\right)+1+\cos 2x-2\cos\left(\frac{\pi}{3}+x\right)\cos x\right)$$

$$=\frac{1}{2}\left(2+\cos\left(\frac{\pi}{3}+2x\right)+\cos 2x+\cos\frac{\pi}{3}-\cos\left(\frac{\pi}{3}+2x\right)\right)$$

$$=\frac{1}{2}\left(2-\frac{1}{2}+2\cos\left(\frac{\pi}{3}+2x\right)\cos\frac{\pi}{3}-\cos\left(\frac{\pi}{3}+2x\right)\right) \text{ since } \cos\frac{\pi}{3}=\frac{1}{2}$$

$$=\frac{3}{4}+\frac{1}{2}\cos\left(\frac{\pi}{3}+2x\right)-\frac{1}{2}\cos\left(\frac{\pi}{3}+2x\right)=\frac{3}{4}$$

and this does not contain x. Hence proved.

11. If $\dfrac{\cos^4 x}{\cos^2 y}+\dfrac{\sin^4 x}{\sin^2 y}=1$, then prove that $\dfrac{\cos^4 y}{\cos^2 x}+\dfrac{\sin^4 y}{\sin^2 x}=1$.

Solution: The given condition is

$$\cos^4 x\sin^2 y+\sin^4 x\cos^2 y=\sin^2 y\cos^2 y$$
$$=\sin^2 y(1-\sin^2 y)=\sin^2 y-\sin^4 y \qquad (1)$$

Therefore,

$$\sin^4 y=\sin^2 y(1-\cos^4 x)-\sin^4 x\cos^2 y$$
$$=\sin^2 y(1-\cos^2 x)(1+\cos^2 x)-\sin^4 x\cos^2 y$$
$$=\sin^2 y\sin^2 x(1+\cos^2 x)-\sin^4 x\cos^2 y$$

Hence,

$$\frac{\sin^4 y}{\sin^2 x}=\sin^2 y+\sin^2 y\cos^2 x-\cos^2 y\sin^2 x \qquad (2)$$

Similarly, on the R.H.S. of Eq. (1), replacing $\sin^2 y$ by $1-\cos^2 y$ and simplifying as shown above, we get

$$\frac{\cos^4 y}{\cos^2 x}=\cos^2 y+\cos^2 y\sin^2 x-\cos^2 x\sin^2 y \qquad (3)$$

By adding Eqs. (2) and (3), we get the desired result.

12. Prove that:

1. $\tan A+\cot A=2\operatorname{cosec}2A$
2. $\cot A-\tan A=2\cot 2A$

Deduce that $\tan A+2\tan 2A+4\tan 4A+8\cot 8A=\cot A$ and more generally
$\tan A+2\tan 2A+2^2\tan 2^2A+\cdots+2^{n-1}\tan^{2n-1}A+2^n\cot 2^nA=\cot A$

Solution:

1. $\tan A+\cot A=\tan A+\dfrac{1}{\tan A}=\dfrac{1+\tan^2 A}{\tan A}$

$$=\frac{\sec^2 A}{\tan A}=\frac{2}{2\tan A\cos^2 A}=\frac{2}{\sin 2A}=2\operatorname{cosec}2A$$

2. $\cot A-\tan A=\dfrac{\cos A}{\sin A}-\dfrac{\sin A}{\cos A}=\dfrac{\cos^2 A-\sin^2 A}{\sin A\cos A}=\dfrac{2\cos 2A}{\sin 2A}=2\cot 2A$

Therefore,

$\tan A=\cot A-2\cot 2A$ (1)
$\tan 2A=\cot 2A-2\cot 4A$ [changing A to $2A$ in Eq. (1)] (2)
$\tan 4A=\cot 4A-2\cot 8A$ [similar change] (3)

Multiplying Eqs. (1) – (3) by 1, 2, 2^2 and adding, we get

$$\tan A+2\tan 2A+2^2\tan 4A=\cot A-8\cot 8A$$

Hence,

$$\tan A+2\tan 2A+2^2\tan 2^2A+2^3\cot 2^3A=\cot A$$

The general result can be obtained by repeating the above sequence of steps n times.

13. If $A+B+C=\pi$, and

$$\tan\left(\frac{A+B-C}{4}\right)\tan\left(\frac{B+C-A}{4}\right)\tan\left(\frac{A+C-B}{4}\right)=1$$

prove that $\sin A+\sin B+\sin C+\sin A\sin B\sin C=0$.

Solution:

$$\tan\left(\frac{A+B-C}{4}\right)=\tan\left(\frac{\pi-2C}{4}\right)=\tan\left(\frac{\pi}{4}-\frac{C}{2}\right)=\frac{1-\tan\frac{C}{2}}{1+\tan\frac{C}{2}}$$

$$=\frac{\left(\cos\frac{C}{2}-\sin\frac{C}{2}\right)^2}{\cos^2\frac{C}{2}-\sin^2\frac{C}{2}}=\frac{1-\sin C}{\cos C}=\frac{\cos C}{1+\sin C}$$

Similarly,

$$\tan\left(\frac{B+C-A}{4}\right)=\frac{1-\sin A}{\cos A}=\frac{\cos A}{1+\sin A}$$

and $$\tan\left(\frac{C+A-B}{4}\right)=\frac{1-\sin B}{\cos B}=\frac{\cos B}{1+\sin B}$$

The given condition implies

$$\left(\frac{1-\sin A}{\cos A}\right)\left(\frac{1-\sin B}{\cos B}\right)\left(\frac{1-\sin C}{\cos C}\right)=1 \qquad (1)$$

as well as

$$\left(\frac{\cos A}{1+\sin A}\right)\left(\frac{\cos B}{1+\sin B}\right)\left(\frac{\cos C}{1+\sin C}\right)=1 \qquad (2)$$

From Eqs. (1) and (2), we get

$$\cos A\cos B\cos C=(1-\sin A)(1-\sin B)(1-\sin C)$$
$$=(1+\sin A)(1+\sin B)(1+\sin C)$$

Hence,

$$1-\Sigma\sin A+\Sigma\sin A\sin B-\sin A\sin B\sin C$$
$$=1+\Sigma\sin A+\Sigma\sin A\sin B+\sin A\sin B\sin C$$

Therefore, $\Sigma\sin A+\sin A\sin B\sin C=0$.

14. If $0 \le \theta \le \frac{\pi}{2}$, prove the inequality $\cos(\sin\theta) > \sin(\cos\theta)$.

Solution:

We have $\sin\theta + \cos\theta = \sqrt{2}\ \sin\left(\theta + \frac{\pi}{4}\right) \le \sqrt{2}$ since the maximum value of $\sin\left(\theta + \frac{\pi}{4}\right) = 1$

But $\sqrt{2} < \pi/2$; ($\sqrt{2}$ is approximately 1.414 and $\pi/2$ is approximately 1.59). Therefore,

$$\sin\theta + \cos\theta < \frac{\pi}{2}$$

$$\Rightarrow \sin\theta < \frac{\pi}{2} - \cos\theta$$

$\cos(\sin\theta) > \cos\left(\frac{\pi}{2} - \cos\theta\right)$ since $\alpha < \beta \Rightarrow \cos\alpha > \cos\beta$

cosine being a decreasing function in first quadrant. That is

$$\cos(\sin\theta) > \sin(\cos\theta)$$

15. If $\tan(\pi/4 + y/2) = \tan^3(\pi/4 + x/2)$, prove that

$$\sin y = \sin x\left(\frac{3+\sin^2 x}{1+3\sin^2 x}\right)$$

Solution:

$$\tan\left(\frac{\pi}{4} + \frac{y}{2}\right) = \left(\frac{1+\tan\frac{y}{2}}{1-\tan\frac{y}{2}}\right)$$

Therefore, from the given condition,

$$\left(\frac{1+\tan\frac{y}{2}}{1-\tan\frac{y}{2}}\right) = \left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right)^3$$

Hence,

$$\left(\frac{\left(1+\tan\frac{y}{2}\right) - \left(1-\tan\frac{y}{2}\right)}{\left(1+\tan\frac{y}{2}\right) + \left(1-\tan\frac{y}{2}\right)}\right) = \left(\frac{\left(1+\tan\frac{x}{2}\right)^3 - \left(1-\tan\frac{x}{2}\right)^3}{\left(1+\tan\frac{x}{2}\right)^3 + \left(1-\tan\frac{x}{2}\right)^3}\right)$$

$$\left(\because \frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a-b}{a+b} = \frac{c-d}{c+d}\right)$$

$$\Rightarrow \tan\frac{y}{2} = \left(\frac{3\tan\frac{x}{2} + \tan^3\frac{x}{2}}{1+3\tan^2\frac{x}{2}}\right)$$

$$\Rightarrow \tan\frac{y}{2} = \left(\frac{3t+t^3}{1+3t^2}\right) \qquad \left(\text{where } \tan\frac{x}{2} = t\right)$$

$$\text{LHS} = \sin y = \frac{2\tan\frac{y}{2}}{1+\tan^2\frac{y}{2}} = \frac{2\left(\frac{3t+t^3}{1+3t^2}\right)}{1+\left(\frac{3t+t^3}{1+3t^2}\right)^2} = \frac{2(3+t^2)(1+3t^2)}{(1+t^2)(1+14t^2+t^4)}$$

$$\left(\cos x = \frac{1-\tan^2 t}{1+\tan^2 t} \Rightarrow t^2 = \frac{1-\cos x}{1+\cos x}\right)$$

$$= \sin x \frac{2\left(3+\left(\frac{1-\cos x}{1+\cos x}\right)\right)\left(1+3\left(\frac{1-\cos x}{1+\cos x}\right)\right)}{\left(1+\left(\frac{1-\cos x}{1+\cos x}\right)\right)\left(1+14\left(\frac{1-\cos x}{1+\cos x}\right)+\left(\frac{1-\cos x}{1+\cos x}\right)^2\right)}$$

$$= \sin x \frac{2(4+2\cos x)(4-2\cos x)}{((1+\cos x)^2 + 14(1+\cos x)(1-\cos x) + (1-\cos x)^2)}$$

$$= \sin x \frac{3+\sin^2 x}{1+3\sin^2 x} = \text{RHS}$$

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