# ST. LAWRENCE HIGH SCHOOL

A JESUIT CHRISTIAN MINORITY INSTITUTION

FOR GOD AND COUNTRY

## STUDY MATERIAL-12

**SUBJECT – MATHEMATICS**

**1st - Term**

**Chapter: Sequence & Series** **Class: XI**

**Topic: Properties of A.P.** **Date: 06.07.2020**

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## Assuming quantities in AP

If terms are given in AP and their sum is known, then the terms must be picked up in following way:

- For three terms $(a-d), a, (a+d)$
- For four terms $(a-3d), (a-d), (a+d), (a+3d)$
- For five terms $(a-2d), (a-d), a, (a+d), (a+2d)$

**Note:** In general, if we take $(2r+1)$ terms in AP, we take them as

$$a-rd, a-(r-1)d, \ldots, a-d, a, a+d, \ldots, a+rd$$

And if we take $2r$ terms in AP, we take them as

$$(a-(2r-1)d), (a-(2r-3)d), \ldots, (a+(2r-3)d), (a+(2r-1)d)$$

**Example 1.** Sum of three numbers in AP is $-3$ and their product is 8. Find the numbers.

**Solution:** Let the three numbers be $(a-d), a, (a+d)$. Given

$$a-d+a+a+d=3 \Rightarrow 3a=-3 \Rightarrow a=-1$$

Given their product is

$$(a-d)a(a+d)=8 \Rightarrow a(a^2-d^2)=8 \Rightarrow (-1)(1-d^2)=8$$
$$\Rightarrow -1+d^2=8 \Rightarrow d^2=9 \Rightarrow d=\pm 3$$

Therefore, the numbers are $-4, -1, 2$ or $2, -1, -4$.

**Example 2.** Find four numbers in AP whose sum is 20 and the sum of whose square is 120.

**Solution:** Let the four numbers be given by $a - 3d$, $a - d$, $a + d$, $a + 3d$. As per the given condition,

$$20 = (a - 3d) + (a - d) + (a + d) + (a + 3d)$$
$$\Rightarrow 4a = 20 \Rightarrow a = 5$$

Also given is the sum of square = 120. So

$$(a - 3d)^2 + (a - d)^2 + (a + d)^2 + (a + 3d)^2 = 120$$
$$\Rightarrow 4a^2 + 20d^2 = 120 \Rightarrow a^2 + 5d^2 = 30$$
$$\Rightarrow 25 + 5d^2 = 30 \Rightarrow 5d^2 = 5 \Rightarrow d = \pm 1$$

Therefore, the numbers are 2, 4, 6, 8 or 8, 6, 4, 2.

## Properties of AP

- If a fixed number is added (subtracted) to each term of a given AP, then the resulting sequence is also an AP with the same common difference as that of the given AP.
- If each term of an AP is multiplied by a fixed number (say $k$) (or divided by a non-zero fixed number), the resulting sequence is also an AP with the common difference multiplied by $k$.
- If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are two APs with common differences $d$ and $d'$, respectively, then $a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots$ is also an AP with the common difference $d + d'$.
- If $a_1, a_2, a_3, \ldots, a_n$ are in AP, then $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \ldots$ and so on.
- If $n$ arithmetic means $a_1, a_2, \ldots, a_n$ are inserted between the numbers $a$ and $b$ then $a_1 + a_2 + a_3 + \cdots + a_n = n\frac{(a+b)}{2}$.

- If the $n^{th}$ term of any sequence is a linear expression in $n$, then the sequence is an AP whose common difference is the coefficient of $n$.
- If the sum of $n$ terms of any sequence is quadratic in $n$, then the sequence is an AP, whose common difference is twice the coefficient of $n^2$.
- If three terms are in AP, then the middle term is called the arithmetic mean (AM) between the other two, i.e. if $a, b, c$ are in AP then $b = \frac{a+c}{2}$ is the AM of $a$ and $c$.
- If $a_1, a_2, ..., a_n$ are $n$ numbers, then the arithmetic mean ($A$) of these numbers is $A = \frac{1}{n}(a_1 + a_2 + a_3 + \cdots + a_n)$.

**Example 3.** If $a^2 + 2bc, b^2 + 2ac, c^2 + 2ab$ are in AP, show that $\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ are in AP.

**Solution:** Given that $a^2 + 2bc$, $b^2 + 2ac$, $c^2 + 2ab$ are in AP. Then $(a^2 + 2bc) - (ab + bc + ca), (b^2 + 2ab) - (ab + ab + ca), (c^2 + 2ab) - (ab + bc + ca)$ are in AP.

So $(a^2 + bc - ab - ca), (b^2 + ca - ab - ab), (c^2 + ab - bc - ca)$ are in AP.

$$\Rightarrow (a-b)(a-c), (b-c)(b-a), (c-a)(c-b) \text{ are in AP.}$$

$$\Rightarrow \frac{-1}{b-c}, \frac{-1}{c-a}, \frac{-1}{a-b} \text{ are in AP.}$$

$$\Rightarrow \frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b} \text{ are in AP.}$$

**Example 4.** If $a_1, a_2, \ldots, a_n$ are in AP ($a_i > 0$ for all $i$), show that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\cdots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

**Solution:**

$$\text{LHS}=\frac{\sqrt{a_1}-\sqrt{a_2}}{a_1-a_2}+\frac{\sqrt{a_2}-\sqrt{a_3}}{a_2-a_3}+\cdots+\frac{\sqrt{a_{n-1}}-\sqrt{a_n}}{a_{n-1}-a_n}$$

If $d$ is the common difference, then

$$\text{LHS}=\frac{\sqrt{a_1}-\sqrt{a_2}}{-d}+\frac{\sqrt{a_2}-\sqrt{a_3}}{-d}+\cdots+\frac{\sqrt{a_{n-1}}-\sqrt{a_n}}{-d}$$

$$=-\frac{1}{d}\left[\sqrt{a_1}-\sqrt{a_2}+\sqrt{a_2}-\sqrt{a_3}+\cdots+\sqrt{a_{n-1}}-\sqrt{a_n}\right]$$

$$=-\frac{1}{d}\frac{(a_1-a_n)}{\sqrt{a_1}+\sqrt{a_n}}=\frac{(a_n-a_1)}{d}\cdot\frac{1}{\sqrt{a_1}+\sqrt{a_n}}$$

$$=\frac{a_1+(n-1)d-a_1}{d}\cdot\frac{1}{\sqrt{a_1}+\sqrt{a_n}}$$

$$=\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}=\text{RHS}$$

**Example 5.** If $a_1, a_2, a_3, \ldots, a_n$ be an AP of non-zero terms, then prove that

$$\frac{1}{a_1a_2}+\frac{1}{a_2a_3}+\cdots+\frac{1}{a_{n-1}a_n}=\frac{n-1}{a_1a_n}$$

**Solution:** Let $d$ be the common difference of the given AP. Then

$$a_2-a_1=a_3-a_2=a_n-a_{n-1}=d \text{ (say)}$$

Now,

$$\frac{1}{a_1a_2}+\frac{1}{a_2a_3}+\frac{1}{a_3a_4}+\cdots+\frac{1}{a_{n-1}a_n}$$

$$= \frac{1}{d}\left[\frac{d}{a_1a_2} + \frac{d}{a_2a_3} + \frac{d}{a_3a_3} + \cdots + \frac{d}{a_{n-1}a_n}\right]$$

$$= \frac{1}{d}\left[\frac{a_2 - a_1}{a_1a_2} + \frac{a_3 - a_2}{a_2a_3} + \cdots + \frac{a_n - a_{n-1}}{a_{n-1}a_n}\right]$$

$$= \frac{1}{d}\left[\frac{1}{a_1} - \frac{1}{a_2} + \frac{1}{a_2} - \frac{1}{a_3} + \cdots + \frac{1}{a_{n-1}} - \frac{1}{a_n}\right]$$

$$= \frac{1}{d}\left[\frac{1}{a_1} - \frac{1}{a_n}\right]$$

$$= \frac{1}{d}\left[\frac{a_n - a_1}{a_1a_n}\right] = \frac{1}{d}\left[\frac{a_1 + (n-1)d - a}{a_1a_n}\right]$$

$$= \frac{n-1}{a_1a_n} = \text{RHS}$$

**Example 6.** Find the sum of first 24 terms of the AP $a_1, a_2, a_3, \ldots$, if it is known that $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$.

**Solution:** As we know in an AP, the sum of the terms equidistant from the beginning and the end is always same and is equal to the sum of first and last terms. Therefore

$$a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \ldots$$

So

$$a_1 + a_{24} = a_5 + a_{20} = a_{10} + a_{15}$$

$$\Rightarrow 3(a_1 + a_{24}) = 225 \Rightarrow a_1 + a_{24} = 75$$

So

$$S_{24} = 24/2\ (a_1 + a_{24}) = 12 \times 75 = 780$$

**Example 7.** If for a sequence $(a_n)$, $S_n = 3\cdot(2^n - 1)$ find its first term.

**Solution:** Given

$$S_n = 3(2^n - 1)$$
$$S_{n-1} = 3(2^{n-1} - 1)$$

So

$$a_n = S_n - S_{n-1} = 3(2^n - 1) - 3(2^{n-1} - 1)$$
$$= 3(2^n - 2^{n-1}) = 3\cdot 2^{n-1}$$

Therefore $a_1 = 3$.

**Example 8.** If for a sequence $(T_n)$, $S_n = 2n^2 + 3n + 1$ find $T_n$, and $T_1$ and $T_2$.

**Solution:** Given

$$S_n = 2n^2 + 3n + 1$$
$$S_{(n-1)} = 2(n-1)^2 + 3(n-1) + 1$$
$$= 2[n^2 - 2n + 1] + 3n - 2$$
$$= 2n^2 - n$$
$$T_n = S_n - S_{n-1} = 2n^2 + 3n + 1 - 2n^2 + n = 4n + 1$$

Hence, $T_1 = 6$ and $T_2 = 9$.

- **Prepared By**

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