ST. LAWRENCE HIGH SCHOOL

A JESUIT CHRISTIAN MINORITY INSTITUTION

FOR GOD AND COUNTRY

STUDY MATERIAL-25

SUBJECT – MATHEMATICS

2nd - Term

Chapter: Algebra **Class: XI**

Topic: Binomial Theorem **Date: 23.11.2020**

Binomial theorem for any positive integer n,

$$(a + b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + ... + {}^nC_{n-1}a.b^{n-1} + {}^nC_nb^n$$

1. The notation $\sum_{k=0}^{n} {}^nC_k\, a^{n-k}b^k$ stands for

${}^nC_0a^nb^0 + {}^nC_1a^{n-1}b^1 + ... + {}^nC_ra^{n-r}b^r + ... + {}^nC_na^{n-n}b^n$, where $b^0 = 1 = a^{n-n}$.

Hence the theorem can also be stated as

$$(a+b)^n = \sum_{k=0}^{n} {}^nC_k\, a^{n-k}b^k .$$

2. The coefficients ${}^nC_r$ occuring in the binomial theorem are known as binomial coefficients.

(i) Taking $a = x$ and $b = -y$, we obtain

$$\begin{aligned}(x - y)^n &= [x + (-y)]^n \\ &= {}^nC_0x^n + {}^nC_1x^{n-1}(-y) + {}^nC_2x^{n-2}(-y)^2 + {}^nC_3x^{n-3}(-y)^3 + ... + {}^nC_n(-y)^n \\ &= {}^nC_0x^n - {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 - {}^nC_3x^{n-3}y^3 + ... + (-1)^n\,{}^nC_n\,y^n\end{aligned}$$

Thus $(x-y)^n = {}^nC_0x^n - {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + ... + (-1)^n\,{}^nC_n\,y^n$

(ii) Taking $a = 1$, $b = x$, we obtain

$$\begin{aligned}(1 + x)^n &= {}^nC_0(1)^n + {}^nC_1(1)^{n-1}x + {}^nC_2(1)^{n-2}x^2 + ... + {}^nC_nx^n \\ &= {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + ... + {}^nC_nx^n\end{aligned}$$

Thus $(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + ... + {}^nC_nx^n$

In particular, for $x = 1$, we have

$$2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n.$$

(iii) Taking $a = 1$, $b = -x$, we obtain

$$(1-x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 - ... + (-1)^n\, {}^nC_n x^n$$

In particular, for $x = 1$, we get

$$0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - ... + (-1)^n\, {}^nC_n$$

Example Expand $\left(x^2 + \frac{3}{x}\right)^4$, $x \neq 0$

Solution By using binomial theorem, we have

$$\left(x^2 + \frac{3}{x}\right)^4 = {}^4C_0 (x^2)^4 + {}^4C_1 (x^2)^3 \left(\frac{3}{x}\right) + {}^4C_2 (x^2)^2 \left(\frac{3}{x}\right)^2 + {}^4C_3 (x^2) \left(\frac{3}{x}\right)^3 + {}^4C_4 \left(\frac{3}{x}\right)^4$$

$$= x^8 + 4.x^6 \cdot \frac{3}{x} + 6.x^4 \cdot \frac{9}{x^2} + 4.x^2 . \frac{27}{x^3} + \frac{81}{x^4}$$

$$= x^8 + 12x^5 + 54x^2 + \frac{108}{x} + \frac{81}{x^4}.$$

Example Compute $(98)^5$.

Solution We express 98 as the sum or difference of two numbers whose powers are easier to calculate, and then use Binomial Theorem.

Write $98 = 100 - 2$

Therefore, $(98)^5 = (100 - 2)^5$

$$= {}^5C_0\,(100)^5 - {}^5C_1\,(100)^4.2 + {}^5C_2\,(100)^3 2^2 - {}^5C_3\,(100)^2\,(2)^3 + {}^5C_4\,(100)\,(2)^4 - {}^5C_5\,(2)^5$$

$$= 10000000000 - 5 \times 100000000 \times 2 + 10 \times 1000000 \times 4 - 10 \times 10000 \times 8 + 5 \times 100 \times 16 - 32$$

$$= 10040008000 - 1000800032 = 9039207968.$$

General and Middle Terms

1. In the binomial expansion for $(a + b)^n$, we observe that the first term is ${}^nC_0a^n$, the second term is ${}^nC_1a^{n-1}b$, the third term is ${}^nC_2a^{n-2}b^2$, and so on. Looking at the pattern of the successive terms we can say that the $(r + 1)^{th}$ term is ${}^nC_ra^{n-r}b^r$. The $(r + 1)^{th}$ term is also called the *general term* of the expansion $(a + b)^n$. It is denoted by T_{r+1}. Thus $T_{r+1} = {}^nC_r\, a^{n-r}b^r$.

2. Regarding the middle term in the expansion $(a + b)^n$, we have

   (i) If n is even, then the number of terms in the expansion will be $n + 1$. Since n is even so $n + 1$ is odd. Therefore, the middle term is $\left(\frac{n+1+1}{2}\right)^{th}$, i.e., $\left(\frac{n}{2}+1\right)^{th}$ term.

   For example, in the expansion of $(x + 2y)^8$, the middle term is $\left(\frac{8}{2}+1\right)^{th}$ i.e., 5th term.

   (ii) If n is odd, then $n + 1$ is even, so there will be two middle terms in the expansion, namely, $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+1}{2}+1\right)^{th}$ term. So in the expansion $(2x - y)^7$, the middle terms are $\left(\frac{7+1}{2}\right)^{th}$, i.e., 4th and $\left(\frac{7+1}{2}+1\right)^{th}$, i.e., 5th term.

Example Show that the middle term in the expansion of $(1+x)^{2n}$ is $\frac{1.3.5...(2n-1)}{n!}2n\,x^n$, where n is a positive integer.

Solution As $2n$ is even, the middle term of the expansion $(1 + x)^{2n}$ is $\left(\frac{2n}{2}+1\right)^{\text{th}}$, i.e., $(n + 1)^{\text{th}}$ term which is given by,

$$T_{n+1} = {}^{2n}C_n(1)^{2n-n}(x)^n = {}^{2n}C_nx^n = \frac{(2n)!}{n!\ n!}x^n$$

$$= \frac{2n(2n-1)\ (2n-2)\ \ldots 4.3.2.1}{n!\ n!} x^n$$

$$= \frac{1.2.3.4\ldots(2n-2)(2n-1)(2n)}{n!n!} x^n$$

$$= \frac{[1.3.5\ldots(2n-1)][2.4.6\ldots(2n)]}{n!n!} x^n$$

$$= \frac{[1.3.5\ldots(2n-1)]2^n[1.2.3\ldots n]}{n!n!} x^n$$

$$= \frac{[1.3.5\ldots(2n-1)]\,n!}{n!\ n!} 2^n.x^n$$

$$= \frac{1.3.5\ldots(2n-1)}{n!} 2^n\ x^n$$

Prepared by :-

Mr. Sukumar Mandal (SkM)