STUDY MATERIAL-17

SUBJECT – MATHEMATICS

1st - Term

Chapter: Algebra **Class: XI**

Topic: Complex numbers (Part 2) **Date: 01.08.2020**

• Introduction :-

Whenever $\sqrt{x}$ is thought to give a real value, it has been, till now, insisted that $x \geq 0$. In other words, in the set of real numbers it is not possible to provide a value for the existence of $\sqrt{x}$ when $x < 0$. To make this possible, we extend the number system so as to include and cover yet another class of numbers called imaginary numbers.

Let us take the quadratic equation, $x^2 - 2x + 10 = 0$. The formal solution of this equation is $\frac{2 \pm \sqrt{4 - 40}}{2}$, that is, $1 \pm 3\sqrt{-1}$, which is not meaningful in the set of real numbers. So, a symbol $i = \sqrt{-1}$ is introduced.

The symbol i, is thought to possess the following properties:

1. It combines with itself and with real numbers satisfying the laws of algebra.
2. Whenever we come across -1 we may substitute i^2.

In the light of the foregoing, the roots of the equation discussed earlier may be taken as $1 + 3i$ and $1 - 3i$.

It is considered that 1 is the real part and 3 (or –3) is the imaginary part of the complex number $1 + 3i$ (or $1 - 3i$).

It has now to be mentioned that the "+" symbol appearing between 1 and $3i$ does not seem to be meaningful, though the following are true:

$$(x_1 + iy_1) \pm (x_2 + iy_2) = (x_1 \pm x_2) + i(y_1 \pm y_2) \tag{5.1}$$

The real parts are added (or subtracted) separately and so in fact are the imaginary parts [Eq. (5.1)].

Also, $$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \tag{5.2}$$

$$\frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + \frac{i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2} \tag{5.3}$$

To make these operations really meaningful, a formal extension of the number system is presented in this lesson.

EXAMPLE 1 If $x = -5 + 2\sqrt{-4}$, then find the value of $x^4 + 9x^3 + 35x^2 - x + 4$.

Solution:

$$x = -5 + 2{\cdot}2\sqrt{-1}$$

$$x = -5 + 4i \; (i = \sqrt{-1})$$

$$x + 5 = 4i$$

Squaring both sides, we get

$$x^2 + 10x + 25 = -16 \Rightarrow x^2 + 10x + 41 = 0$$

Now,

$$x^4+9x^3+35x^2-x+4=(x^2+10x+41)\,(x^2-x+4)-160$$

We know,

$$x^2+10x+41=0$$

$$\Rightarrow x^4+9x^3+35x^2-x+4=0-160=-160$$

Hence, the value of given expression is -160.

• Complex Numbers :-

A complex number, represented by an expression in the form $x+iy$ (where x, y are the real numbers), is considered to be an ordered pair (x, y) of two real numbers, combined to form a complex number, and an algebra is defined in the set of such numbers, represented by an ordered pair (x, y) to satisfy the following:

(addition) $(x_1, y_1)+(x_2, y_2)=(x_1+x_2, y_1+y_2)$

(subtraction) $(x_1, y_1)-(x_2, y_2)=(x_1-x_2, y_1-y_2)$

(multiplication) $(x_1, y_1)\times(x_2, y_2)=(x_1x_2-y_1y_2, x_1y_2+x_2y_1)$

(division) $(x_1, y_1)\div(x_2, y_2)=\left(\dfrac{x_1x_2+y_1y_2}{x_2^2+y_2^2}, \dfrac{x_2y_1-x_1y_2}{x_2^2+y_2^2}\right)$

For any real number α, $\alpha(x, y)=(\alpha x, \alpha y)$ and if $(x, y)=(x', y')$, then it must be $x'=x, y'=y$. In other words, the representation of a complex number in the form (x, y) has a uniqueness property; and for a complex number, it is not possible to have two different forms of the representation of the ordered pairs. In the light of the foregoing, it may be stated that the two representations (x, y) – in the ordered pair form – and $x+iy$ are indistinguishable.

EXAMPLE 2 Find the sum and product of the two complex numbers $Z_1 = 2 + 3i$ and $Z_2 = -1 + 5i$.

Solution:

$$Z_1 + Z_2 = 2 + 3i + (-1 + 5i) = 2 - 1 + 8i = 1 + 8i$$

$$Z_1 Z_2 = (2 + 3i)(-1 + 5i) = -2 + 15i^2 - 3i + 10i = -17 + 7i \quad (i^2 = -1)$$

Based on the above discussion, the following cases have been observed:

1. If $z = a + ib$, then the real part of $z = \text{Re}(z) = a$ and the imaginary part of $z = \text{Im}(z) = b$.
2. If $\text{Re}(z) = 0$, then the complex number is purely imaginary.
3. If $\text{Im}(z) = 0$, then the complex number is real.
4. The complex number $0 = 0 + 0i$ is both purely imaginary and real.
5. Two complex numbers are equal if and only if their real parts and imaginary parts are separately equal, that is, $a + ib = c + id \Leftrightarrow a = c$ and $b = d$.
6. There is no order relation between complex numbers, that is, $(a + ib) >$ or $< (c + id)$ is a meaningless expression.

EXAMPLE 3 Express $\dfrac{1}{(1 - \cos\theta + i\sin\theta)}$ in the form $a + ib$.

Solution:

$$\frac{1}{(1 - \cos\theta + i\sin\theta)} = \frac{(1 - \cos\theta) - i\sin\theta}{(1 - \cos\theta + i\sin\theta)(1 - \cos\theta - i\sin\theta)}$$

$$= \frac{\{(1 - \cos\theta) - i\sin\theta\}}{\{(1 - \cos\theta)^2 + \sin^2\theta\}} = \frac{(1 - \cos\theta) - i\sin\theta}{2 - 2\cos\theta}$$

$$= \frac{1 - \cos\theta}{2(1 - \cos\theta)} - \frac{i \cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \frac{1}{2} - i.\cot\frac{\theta}{2}$$

• Representation of Complex Numbers :-

1. **Geometrical representation:** It is known, from the coordinate geometry, that the ordered pair (x, y) represents a point in the Cartesian plane.
   It is now seen that the ordered pair (x, y) considered as Z represents a complex number.
   It is therefore observed that to every complex number $Z \equiv (x, y)$, one can associate, a point $P \equiv (x, y)$ in the Cartesian plane. The point may be called to be a geometrical representation of Z. This association is a bijection – in the mapping language – whereby the correspondence between Z and P is ONE–ONE and ONTO. It is therefore possible to go over to a point from Z, or reversing the roles, come back to Z from the point.

2. **Argand diagram:** The graphical representation of a complex number $Z = (x, y)$ by a point $P(x, y)$ is called representation in the Argand diagram, which is also called Gaussian plane. In this representation, all complex numbers such as (2, 0), (3, 0), (–1, 0), $(\alpha, 0)$ with the imaginary part 0 will be represented by points on the x-axis. Since the real number α is represented as a complex number $(\alpha, 0)$, all real numbers will get marked on the x-axis. For this reason, the x-axis is called the real axis. Similarly, all purely imaginary numbers (with the real part 0) such as (0, 1), (0, 2), (0, –3), $(0, \beta)$ will be marked on the y-axis. Hence, the y-axis is also called the imaginary axis in this context. The Cartesian plane (two-dimensional plane) is also called the complex plane.

3. **Polar representation:** See Fig. 5.1. Let $P(x, y)$ be any point on the complex plane representing the complex number $z = (x, y)$, with $X'OX$ and $Y'OY$ as the axes of coordinates.

Let $OP = r$ and $\angle XOP = \theta$ (measured anticlockwise).
Then from ΔOMP, we find that

$$x = OM = r\cos\theta$$

and

$$y = MP = r\sin\theta$$

Thus,

$$z = (x, y) = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

Also,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta \text{ by Euler's formula}$$

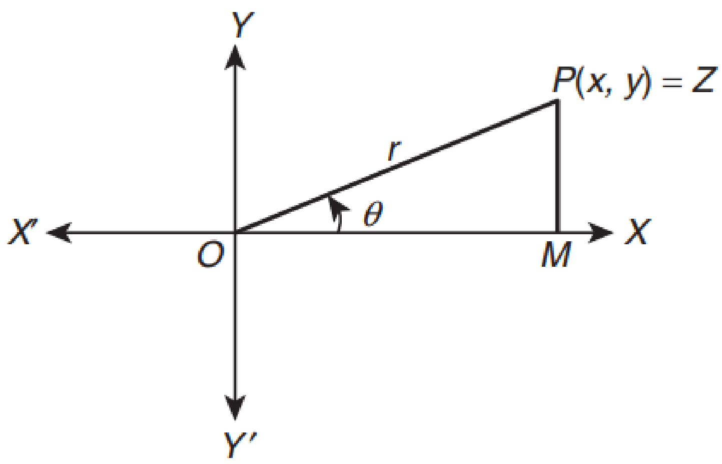

Figure 5.1

Thus, $z = r(\cos\theta + i\sin\theta)$ can be written as

$$z = re^{i\theta}$$

This form of representation of Z is called the *trigonometric form* or the *polar form* or the *modulus amplitude form*.
When z is written in the form $r(\cos\theta + i\sin\theta)$, r is called the modulus of z and is written as $|z|$, where

$$|z| = r = \sqrt{x^2 + y^2}$$

a non-negative number. $|z| = 0$ for the only number (0, 0).

EXAMPLE 4 Represent the given complex numbers in the polar form:

$$(1+i\sqrt{3})^2/4i(1-i\sqrt{3})$$

Solution:

$$i(1-i\sqrt{3})=i-i^2\sqrt{3}=\sqrt{3}+i$$

Therefore,

$$\frac{(1+i\sqrt{3})^2}{4i\,(1-i\sqrt{3})}=\frac{(1+i\sqrt{3})^2}{4(\sqrt{3}+i)}=\frac{-2+2i\sqrt{3}}{4(\sqrt{3}+i)}=\frac{(-1+i\sqrt{3})(\sqrt{3}-i)}{2(\sqrt{3}+i)(\sqrt{3}-i)}$$

$$=\frac{-\sqrt{3}+\sqrt{3}+4i}{2(3+1)}=\frac{i}{2}$$

Now,

$$\frac{i}{2}\Rightarrow a+ib$$

$$\Rightarrow a=0,\ b=\frac{1}{2}$$

$$a=r\cos\theta,\ b=r\sin\theta$$

$$\Rightarrow 0=r\cos\theta,\ \frac{1}{2}=r\sin\theta$$

$$\Rightarrow \theta=\frac{\pi}{2},r=\frac{1}{2}$$

So,

$$\frac{i}{2}=\frac{1}{2}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$$

Hence,

$$\frac{(1+i\sqrt{3})^2}{4i\,(1-i\sqrt{3})}=\frac{1}{2}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=\frac{1}{2}e^{i\pi/2}$$

Prepared by :-

Mr. SUKUMAR MANDAL (SkM)