ST. LAWRENCE HIGH SCHOOL

A JESUIT CHRISTIAN MINORITY INSTITUTION

STUDY MATERIAL-26

SUBJECT – MATHEMATICS

Pre-Test

Chapter: Integration **Class: XII**

Topic: Special Forms **Date: 10.07.2020**

1. $\int \sin^m x \cdot \cos^n x \, dx, (m, n \in N)$

If one out of m and n is odd, then substitute for term of even power.
If both are odd, then substitute either of the term.
If both are even, then use trigonometric identities only.

The above substitution enables us to integrate any function of the form $R(\sin x, \cos x)$. However, in practice, it sometimes leads to extremely complex rational function. In some cases, the integral can be simplified by:

(i) Substituting $\sin x = t$, if the integral is of the form $\int R(\sin x)\cos x \, dx$.

(ii) Substituting $\cos x = t$, if the integral is of the form $\int R(\cos x)\sin x \, dx$.

(iii) Substituting $\tan x = t$, that is, $dx = \frac{dt}{1+t^2}$, if the integral is dependent only on $\tan x$.

(iv) Substituting $\cos x = t$, if $R(-\sin x, \cos x) = -R(\sin x, \cos x)$.

(v) Substituting $\sin x = t$, if $R(\sin x, -\cos x) = -R(\sin x, \cos x)$.

(vi) Substituting $\tan x = t$, if $R(-\sin x, -\cos x) = -R(\sin x, \cos x)$.

Example 1. Evaluate $\int \sin^3 x \cdot \cos^2 x \, dx$.

Solution:

$$I = \int \sin x \cdot (1 - \cos^2 x) \cdot \cos^2 x \, dx$$

Put $\cos x = t$. Then $-\sin x \, dx = dt$.

$$I = -\int (t^2 - t^4) dt = \frac{t^5}{5} - \frac{t^3}{3} + c$$

$$= \frac{\sin^5 x}{5} - \frac{\sin^3 x}{3} + c$$

Example 2. Evaluate $\int \frac{d\theta}{\sin\theta \cdot \cos^3\theta}$.

Solution:

$$I = \int \frac{d\theta}{\sin\theta \cdot \cos^3\theta} = \int \frac{\sec^2\theta}{\sin\theta \cdot \cos\theta} d\theta = \int \frac{\sec^2\theta}{\tan\theta \cdot \cos^2\theta} d\theta$$

Put $\tan\theta = t$. Then $\sec^2\theta \, d\theta = dt$.

$$I = \int \frac{1 + t^2}{t} dt = \int \left(\frac{1}{t} + t \right) dt$$

$$I = \ln|t| + \frac{t^2}{2} + c = \ln|\tan\theta| + \frac{\tan^2\theta}{2} + c$$

Example 3. Evaluate $\int \frac{\sin^3 2x}{\cos^5 2x} dx.$

Solution:

$$I = \int \frac{\sin^3 2x}{\cos^5 2x} dx$$

The given equation may be written as

$$I = \int \frac{\sin^3 2x \cdot \sec^2 2x}{\cos^3 2x} dx = \int \tan^3 2x \cdot \sec^2 2x \, dx$$

Put $\tan 2x = t$. Then $2\sec^2 2x \, dx = dt$.

$$I = \frac{1}{2}\int t^3 dt = \frac{t^4}{8} + c = \frac{\tan^4 2x}{8} + c$$

Example 4. Evaluate $\int \frac{\sin nx}{\sin x} dx.$

Solution:

$$I_n = \int \frac{\sin nx}{\sin x} dx$$

$$I_{n-2} = \int \frac{\sin(n-2)x}{\sin x} dx$$

$$I_n - I_{n-2} = \int \frac{\sin nx - \sin(n-2)x}{\sin x} dx = \int \frac{2\cos(n-1)x \cdot \sin x}{\sin x} dx$$

$$I_n - I_{n-2} = \frac{2\sin(n-1)x}{(n-1)}$$

2. Integrals of the form: $\int x^m (a+bx^n)^p \, dx$

(a) If $p \in N$ (natural number). We expand the integral with the help of binomial theorem and integrate.

Example 5. Evaluate $\int x^{1/3} (2+x^{1/2})^2 \, dx$.

Solution:

$$I = \int x^{1/3} (2+x^{1/2})^2 \, dx$$

Since P is a natural number. So

$$I = \int x^{1/3} (4+x+4x^{1/2}) dx$$

$$= \int (4x^{1/3} + x^{4/3} + 4x^{5/6}) dx$$

$$= \frac{4x^{4/3}}{4/3} + \frac{x^{7/3}}{7/3} + \frac{4x^{11/6}}{11/6} + c$$

$$I = 3x^{4/3} + \frac{3}{7} x^{7/3} + \frac{24}{11} x^{11/6} + c$$

(b) If $p \in I^-$ (that is negative integer). Write $x = t^k$ where k is the LCM of the denominator of m and n.

Example 6. Evaluate $\int x^{-2/3}(2+x^{2/3})^{-1}\,dx$.

Solution: If we substitute $x = t^3$ (as we know $p \in$ negative integer), then

$$x = t^3 \Rightarrow dx = 3t^2 dt$$

$$I = \int \frac{3t^2}{t^2(1+t^2)} dt = 3\int \frac{dt}{(1+t^2)} = 3\tan^{-1} t + c$$

$$I = 3\tan^{-1}(x^{1/3}) + c$$

(c) If $\frac{m+1}{n}$ is an integer and $p \in$ fraction, then put $(a+bx^n) = t^k$, where k is the denominator of the fraction p.

(d) If $\left(\frac{m+1}{n} + p\right)$ is an integer and $p \in$ fraction, then put $(a+bx^n) = t^k \cdot x^n$, where k is the denominator of the fraction P.

Example 7. Evaluate $\int x^{-11}(1+x^4)^{-1/2}\,dx.$

Solution: Here,

$$\left(\frac{m+1}{n}+p\right)=\left(\frac{-11+1}{4}+\frac{1}{2}\right)=-3$$

If we substitute $1+\frac{1}{x^4}=t^2$, then

$$-\frac{4}{x^5}dx=2tdt$$

$$I=\int\frac{1}{x^{11}(1+x^4)^{1/2}}dx=\int\frac{1}{x^{11}\cdot x^2(1+x^{-4})^{1/2}}dx$$

$$I=-\frac{1}{4}\int\frac{2t}{x^8 t}dt=-\frac{1}{2}\int(t^2-1)^2\,dt=-\frac{1}{2}\int(t^4-2t^2+1)dt$$

$$I=-\frac{1}{2}\left(\frac{t^5}{5}-2\frac{t^3}{3}+t\right)+c$$

$$=-\frac{1}{2}\left[\frac{\left(1+x^{-4}\right)^{5/2}}{5}-\frac{2}{3}\left(1+x^{-4}\right)^{3/2}+\left(1+x^{-4}\right)^{1/2}\right]+c$$

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