ST. LAWRENCE HIGH SCHOOL

A JESUIT CHRISTIAN MINORITY INSTITUTION

STUDY MATERIAL-3

SUBJECT – MATHEMATICS

Pre-test

Chapter: MATRICES AND DETERMINANTS **Class:** XII

Topic: MATRICES **Date:** 11.05.2020

PART 3

HINTS AND SOLUTIONS—EXERCISE SET 1

1. We have

$$A(\alpha, \beta)^{-1} = \frac{1}{e^{\beta}}\begin{bmatrix} e^{\beta}\cos\alpha & -e^{\beta}\sin\alpha & 0 \\ e^{\beta}\sin\alpha & e^{\beta}\cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = A(-\alpha, -\beta)$$

2. Since $\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ is a square root of I_2, i.e. two-rowed unit matrix

$$\therefore \quad \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha^2 + \beta\gamma & \alpha\beta - \alpha\beta \\ \gamma\alpha - \gamma\alpha & \gamma\beta + \alpha^2 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha^2 + \beta\gamma & 0 \\ 0 & \gamma\beta + \alpha^2 \end{bmatrix}$$

$\therefore \quad \alpha^2 + \beta\gamma = 1$ [by law of equality]

$\Rightarrow \quad 1 - \alpha^2 - \beta\gamma = 0$

3. $|A| = 2(\sin^2 2x - \cos^2 2x) = -2\cos 4x$

4. Given $A^2 + A + 2I = 0$

$\Rightarrow \quad A^2 + A = -2I \quad \Rightarrow \quad |A^2 + A| = |-2I|$

$\Rightarrow \quad |A||A+I| = (-2)^n \quad \Rightarrow \quad |A| \neq 0$

$\Rightarrow A$ is non-singular; hence, its inverse exists. Also multiplying both sides of the given equation with A^{-1}, we get

$$A^{-1} = -\frac{1}{2}(A+I)$$

5. Let $S = E(\theta)E(\phi)$

$$\Rightarrow S = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

$$\Rightarrow S = \begin{bmatrix} \cos\theta\cos\phi\cos(\theta-\phi) & \cos\theta\sin\phi\cos(\theta-\phi) \\ \cos\phi\sin\theta\cos(\theta-\phi) & \sin\theta\sin\phi\cos(\theta-\phi) \end{bmatrix}$$

Since θ and ϕ differ by an odd multiple of $\pi/2$,

$$S = \begin{bmatrix} \cos\theta\cos\phi\cos(2n+1)\frac{\pi}{2} & \cos\theta\sin\phi\cos(2n+1)\frac{\pi}{2} \\ \cos\phi\sin\theta\cos(2n+1)\frac{\pi}{2} & \sin\theta\sin\phi\cos(2n+1)\frac{\pi}{2} \end{bmatrix}$$

$$\therefore \quad S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

6. By definition, any row of a matrix/determinant A when multiplied by its corresponding cofactors results in the value of $|A|$.

7. $$AB = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O_3$$

Short cut method: $|AB| = |A||B| = 0$

As A is skew-symmetric of order 3 so $|A| = 0$, whatever be the value of $|B|$.

$$\therefore \quad |AB| = 0$$

8. $(A+B)^2 = A^2 + AB + BA + B^2 = A + 0 + 0 + B = A + B$

$(A-B)^2 = A + B \neq A - B$

9. We know that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Given $A = \begin{bmatrix} 2x & 0 \\ x & x \end{bmatrix}$

$$\therefore \quad A^{-1} = \frac{1}{2x^2}\begin{bmatrix} x & 0 \\ -x & 2x \end{bmatrix} = \begin{bmatrix} 1/2x & 0 \\ -(1/2x) & 1/x \end{bmatrix}$$

But $A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$

$$\therefore \quad \begin{bmatrix} 1/2x & 0 \\ -(1/2x) & 1/x \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

$$\Rightarrow \quad \frac{1}{2x} = 1 \Rightarrow x = \frac{1}{2}$$

10. $2x + y = 4$ and $3x + 2y = 2$ gives

$$x = \frac{\begin{vmatrix} 4 & 1 \\ 2 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}} = \frac{6}{1} = 6\,;\; y = \frac{\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix}}{1} = \frac{-8}{1} = -8$$

Clearly, $x + y = -2$ is also satisfied.

Hence, the given system has one solution: $x = 6$, $y = -8$.

11. $A^{-1} = \frac{1}{|A|}(\text{adj}A)$ using the fact $|A|A^{-1} = \text{adj}A$

$\therefore \quad 13A^{-1} = \text{adj}A$

12. $P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}, A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, Q = PAP^T$

Here, $P \cdot P^T = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \quad P \cdot P^T = I$

Now, $Q = PAT^T$. Also, $x = P^T Q^{2005} P$

$\Rightarrow \quad x = P^T (PAP^T)^{2005} P$

$\Rightarrow \quad x = P^T (PAP^T)(PAP^T)^{2004} P$

$\Rightarrow \quad x = (IA)P^T (PAP^T)^{2004} P$

$\Rightarrow \quad x = AP^T (PAP^T)(PAP^T)^{2003} P \ldots$

$\Rightarrow \quad x = A^{2004} P^T (PAP^T) P$

$\Rightarrow \quad x = A^{2004} (P^T P) A (P^T P) = A^{2004} IAI$

$\therefore \quad x = A^{2005} = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$

13. $f(x) = \begin{bmatrix} \sin x & \text{cosec } x & \tan x \\ \sec x & x\sin x & x\tan x \\ x^2-1 & \cos x & x^2+1 \end{bmatrix}$, then

$$|f(x)| = \begin{vmatrix} \sin x & \text{cosec } x & \tan x \\ \sec x & x\sin x & x\tan x \\ x^2-1 & \cos x & x^2+1 \end{vmatrix} = g(x) \quad \text{(assume)}$$

so $g(-x)=\begin{vmatrix}-\sin x & -\text{cosec } x & -\tan x\\ \sec x & x\sin x & x\tan x\\ x^2-1 & \cos x & x^2+1\end{vmatrix}=-g(x)$

$\therefore\quad g(-x)=-g(x)$

$\Rightarrow\quad g(x)=|f(x)|$ is an odd function.

$\therefore\quad \int_{-a}^{a}|f(x)|\,dx=0.$

14. Since $A=\begin{bmatrix}\alpha & 0\\ 1 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\ 5 & 1\end{bmatrix}$

Also, $A^2=B$

$\Rightarrow\begin{bmatrix}\alpha & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}\alpha & 0\\ 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 5 & 1\end{bmatrix}\Rightarrow\begin{bmatrix}\alpha^2+1 & 0\\ \alpha+1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 5 & 1\end{bmatrix}$

$\therefore\quad \alpha^2+1=1 \quad\Rightarrow\quad \alpha=0$

$\Rightarrow\quad \alpha+1=5$ or $\alpha=4$

Both the values of α are not possible simultaneously.

$\therefore$ There is no real value of α for which $A^2=B$.

15. $A=\begin{bmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{bmatrix}=4I_3$

$\therefore\quad A^n=4^n(I_3)^n=2^{2n}I_3$

$$\Rightarrow\quad \begin{bmatrix}2^{2n} & 0 & 0\\ 0 & 2^{2n} & 0\\ 0 & 0 & 2^{2n}\end{bmatrix}$$

16. Since $A(\text{adj}\,A)=|A|I$

$$\Rightarrow\quad \begin{bmatrix}10 & 0\\ 0 & 10\end{bmatrix}=|A|\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}|A| & 0\\ 0 & |A|\end{bmatrix}$$

$\therefore\quad |A|=10$

17. $A=\begin{bmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{bmatrix}$, where $t=\dfrac{2\pi}{3}$

$\therefore\quad 3t=2\pi$

$\therefore\quad A^2=\begin{bmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{bmatrix}\begin{bmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{bmatrix}$

$A^2=\begin{bmatrix}\cos 2t & -\sin 2t\\ \sin 2t & \cos 2t\end{bmatrix}$

$\Rightarrow\quad A^2\cdot A=A^3=\begin{bmatrix}\cos 2t & -\sin 2t\\ \sin 2t & \cos 2t\end{bmatrix}\begin{bmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{bmatrix}$

$\Rightarrow\quad \begin{bmatrix}\cos 3t & -\sin 3t\\ \sin 3t & \cos 3t\end{bmatrix}=\begin{bmatrix}\cos 2\pi & -\sin 2\pi\\ \sin 2\pi & \cos 2\pi\end{bmatrix}$

$\Rightarrow\quad A^3=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$\therefore$ The least value of $k=3$.

18. We know that every identity matrix is a scalar matrix. Therefore, option (a) is correct.

19. For the system of equations to have non-trivial solution,

$$\Delta=\begin{vmatrix}t & t+1 & t-1\\ t+1 & t & t+2\\ t-1 & t+2 & t\end{vmatrix}=0$$

Applying $C_2\to C_2-C_1, C_3\to C_3-C_1$, we get

$$\Delta=\begin{vmatrix}t & 1 & -1\\ t+1 & -1 & 1\\ t-1 & 3 & 1\end{vmatrix}=\begin{vmatrix}2t+1 & 0 & 0\\ t+1 & -1 & 1\\ t-1 & 3 & 1\end{vmatrix}=0$$

[using $R_1\to R_1+R_2$]

$=(2t+1)(-4)=0$

$\Rightarrow\quad t=-\dfrac{1}{2}.$

20. We know that $A(\text{adj}A)=|A|I$.

If $A=\begin{bmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{bmatrix}$, then $|A|=1$

$\therefore\quad A(\text{adj}A)=I.$

It is given that $A(\text{adj}A)=kI$. Therefore, $k=1$.

21. We know that $\text{adj}(\text{adj}A)=|A|^{n-2}A$ if $A\neq 0$, provided the order of A is n.

$\Rightarrow\quad \text{adj}(\text{adj}A)=|A|A\qquad$ (as $n=3$)

$\therefore\quad \det(\text{adj}(\text{adj}A))=|A|^3\det A=|A|^4$

But $|A|=\begin{bmatrix}1 & 2 & -1\\ -1 & 1 & 2\\ 2 & -1 & 1\end{bmatrix}=14$

$\therefore\quad \det[\text{adj}(\text{adj}A)]=(14)^4.$

22. Given square matrices A and B are of the same order. We know that if A and B are square matrices then $(AB)^{-1}=B^{-1}A^{-1}$.

23. If $p=q=r$, then $x=0, y=0, z=0$ is a solution of the system of equations.
If a, b, c are distinct, then the determinant of the coefficient matrix is

$$\frac{1}{2}(a+b+c)[(b-c)^2+(c-a)^2+(a-b)^2]\neq 0$$

and hence the system of equations has a unique solution.

24. We have $3A^3 + 2A^2 + 5A + I = 0$

$\Rightarrow \quad I = -3A^3 - 2A^2 - 5A$

$\Rightarrow \quad IA^{-1} = (-3A^3 - 2A^2 - 5A)A^{-1}$

$\Rightarrow \quad A^{-1} = -3A^2 - 2A - 5I$

25. Since the system possesses no solution

$\therefore \quad \Delta = \begin{vmatrix} 2 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$

$\Rightarrow \quad \lambda = 1$

26. We know that $AB = AC$ which implies that $B = C$. Therefore, the option (c) is not true for the matrices.

27. We have $(A + B)(A - B) = A^2 - AB + BA - B^2$.
So, (a) is not true.

28. Given square matrix $A = [a_{ij}]$ and $a_{ij} = a_{ji}$.
We know that in a square matrix if $a_{ij} = a_{ji}\ \forall i, j$, then ith, jth elements are equal to jth, ith elements in matrix A. Therefore, A is a symmetric matrix.

29. Since A is a non-singular matrix, therefore $|A| \neq 0$

$\Rightarrow$ rank $(A) = n$.

30. Given order of the square matrices A and B, $n = 3$, $|A| = -1$ and $|B| = 3$. We know that if A and B are two square matrices of order n, then $|rAB| = r^n |A||B|$. Comparing $|rAB|$ to the given matrix $|3AB|$, we get $r = 3$. Therefore, $|3AB| = (3)^3 |A||B| = 27(-1)(3) = -81$.

31. We have $I_3 = AB = \begin{bmatrix} 1 & 2 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & x+y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \quad x + y = 0$

32. Given matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and the matrix equation

$$A^2 - kA - I_2 = 0 \Rightarrow kA = A^2 - I_2 \quad \text{(i)}$$

We know that $A^2 = A \cdot A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix}$

and $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Substituting the values of A^2, A and I_2 in Eq. (i), we get

$$k\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 8 & 12 \end{bmatrix}$$

$\Rightarrow \quad k\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = 4\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.

Comparing both sides, we get $k = 4$.

33. Since A satisfies the equation $x^3 - 5x^2 + 4x + k = 0$, $A^3 - 5A^2 + 4A + kI = 0$.

$\Rightarrow \quad A^{-1}$ exists if $k \neq 0$.

34. Matrix $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and matrix equation $|A^3| = 125$.

Since $|A|^3 = 125$, therefore $|A| = 5$. We know that

$$|A| = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix} = \alpha^2 - 4$$

Substituting the value of $|A|$ in the above equation, we get

$\Rightarrow \quad 5 = \alpha^2 - 4 \quad \Rightarrow \quad \alpha^2 = 9 \quad \Rightarrow \quad \alpha = \pm 3$.

35. Number of columns in A = Number of rows in B. Hence B is a 3×5 matrix.

36. $AB = A$

$\Rightarrow \quad ABA = A.A = A^2$

$\Rightarrow \quad A(BA) = A^2$

$\Rightarrow \quad AB = A^2 \quad [\because\ BA = B]$

$\Rightarrow \quad A = A^2 \quad [\because\ AB = A]$

Thus, $A^2 = A$

37. We have $A^2 = AA$

$\Rightarrow \quad A^2 = (AB)A \quad [\because\ AB = A]$

$\Rightarrow \quad A^2 = A(BA)$

$\Rightarrow \quad A^2 = AB \quad [\because\ BA = B]$

$\Rightarrow \quad A^2 = A \quad [\because\ AB = A]$

and $B^2 = BB$

$\Rightarrow \quad B^2 = (BA)B \quad [\because\ BA = B]$

$\Rightarrow \quad B^2 = B(AB)$

$\Rightarrow \quad B^2 = BA \quad [\because\ AB = A]$

$\Rightarrow \quad B^2 = B \quad [\because\ BA = B]$

38. Since AB exists, therefore the number of columns in A = the number of rows in B. So, B has n rows. Since BA exists, the number of columns in B = the number of rows in A. So, B has m columns.

$\therefore \quad B$ is an $n \times m$ matrix.

39. For the given matrix to be a zero matrix, each entry in the matrix must be zero, which is possible when $x = y = z = a = b = c$.

40. Matrix $A = \begin{bmatrix} 6 & x-2 \\ 3 & x \end{bmatrix}$.

We know that if $|A| = 0$, then A has no inverse.

Therefore, $|A| = 6x - 3x + 6 = 0$ or $x = -2$.

41. $x+y=5, x-y=3$

$\Rightarrow \quad x=4, y=1$

42. We have $A=\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$

$\therefore \quad A^2=\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}$

$A^3=\begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\3 & 1\end{bmatrix}$

By induction, $A^n=\begin{bmatrix}1 & 0\\n & 1\end{bmatrix}$ (i)

$\therefore \quad nA-(n-1)I=n\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}-(n-1)\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

$=\begin{bmatrix}n & 0\\n & n\end{bmatrix}-\begin{bmatrix}(n-1) & 0\\0 & (n-1)\end{bmatrix}$

$=\begin{bmatrix}1 & 0\\n & 1\end{bmatrix}=A^n.$ [using (i)]

43. We have $A=\begin{bmatrix}i & 0\\0 & i\end{bmatrix}$

Clearly, A is a diagonal matrix. Therefore, A^{4n} is also a diagonal matrix such that

$$A^{4n}=\begin{bmatrix}i^{4n} & 0\\0 & i^{4n}\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

44. $(ABA)^T=A^TB^TA^T=ABA$

$\Rightarrow \quad ABA$ is symmetric.

45. Matrix $\begin{bmatrix}a & a^2 & a^3-1\\b & b^2 & b^3-1\\c & c^2 & c^3-1\end{bmatrix}=0$

The given matrix may be written as

$$\begin{bmatrix}a & a^2 & a^3\\b & b^2 & b^3\\c & c^2 & c^3\end{bmatrix}+\begin{bmatrix}a & a^2 & -1\\b & b^2 & -1\\c & c^2 & -1\end{bmatrix}=0$$

$$\Rightarrow \quad abc\begin{bmatrix}1 & a & a^2\\1 & b & b^2\\1 & c & c^2\end{bmatrix}-\begin{bmatrix}1 & a & a^2\\1 & b & b^2\\1 & c & c^2\end{bmatrix}=0$$

$$\Rightarrow \quad (abc-1)\begin{bmatrix}1 & a & a^2\\1 & b & b^2\\1 & c & c^2\end{bmatrix}=0$$

$\Rightarrow \quad abc=1$

46. $2A+3B-C^T$

$=\begin{bmatrix}4 & 6 & 2\\8 & -4 & 12\end{bmatrix}+\begin{bmatrix}0 & 9 & 6\\-6 & 3 & 12\end{bmatrix}-\begin{bmatrix}7 & 9 & 6\\8 & 0 & 5\end{bmatrix}$

$=\begin{bmatrix}-3 & 6 & 2\\-6 & -1 & 19\end{bmatrix}$

47. $A^2-A+I=0 \quad \Rightarrow \quad A^2-A=-I$

$\Rightarrow \quad A(A-I)=-I \quad \Rightarrow \quad A(I-A)=I.$

Hence, $A^{-1}=I-A$.

48. We have

$$A=\omega\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=\omega I_2$$

$\therefore \quad A^{100}=\omega^{100}(I_2)^{100}=\omega^{100}I_2=\omega I_2=A$

49. If $A=\begin{bmatrix}d_1 & 0\\0 & d_2\end{bmatrix}$, then

$$A^2=\begin{bmatrix}d_1 & 0\\0 & d_2\end{bmatrix}\begin{bmatrix}d_1 & 0\\0 & d_2\end{bmatrix}=\begin{bmatrix}d_1^2 & 0\\0 & d_2^2\end{bmatrix}.$$

Hence the option (c) is correct.

50. $\begin{bmatrix}1 & x & 1\end{bmatrix}\begin{bmatrix}2 & 3 & 2\\0 & 5 & 1\\0 & 3 & 2\end{bmatrix}\begin{bmatrix}1\\1\\x\end{bmatrix}=0$

$\Rightarrow \quad \begin{bmatrix}2 & 6+5x & 4+x\end{bmatrix}\begin{bmatrix}1\\1\\x\end{bmatrix}=0$

$\Rightarrow \quad 2+(6+5x)+(4+x)x=0 \quad \Rightarrow \quad x^2+9x+8=0$

$\Rightarrow \quad (x+1)(x+8)=0 \quad \Rightarrow \quad x=-1,-8$

51. Here $A=\begin{bmatrix}1 & -1 & 1\\2 & 1 & -3\\1 & 1 & 1\end{bmatrix}$

$\therefore \quad |A|=\begin{vmatrix}1 & -1 & 1\\2 & 1 & -3\\1 & 1 & 1\end{vmatrix}=1\cdot(1+3)+1\cdot(2+3)+1\cdot(2-1)$

$=4+5+1=10$

$\therefore \quad \text{adj}A=\begin{bmatrix}4 & -5 & 1\\2 & 0 & -2\\2 & 5 & 3\end{bmatrix}^T=\begin{bmatrix}4 & 2 & 2\\-5 & 0 & 5\\1 & -2 & 3\end{bmatrix}$

$\therefore \quad B=A^{-1}=\frac{1}{10}\begin{bmatrix}4 & 2 & 2\\-5 & 0 & 5\\1 & -2 & 3\end{bmatrix} \Rightarrow (10)B=\begin{bmatrix}4 & 2 & 2\\-5 & 0 & 5\\1 & -2 & 3\end{bmatrix}.$

Hence $\alpha=5$.

52. We have

$$(I-A)(I+A+A^2)=I+A+A^2-A-A^2-A^3$$
$$=I-0=I$$

$\therefore \quad (I-A)^{-1}=I+A+A^2$

53. $AA^{-1}=I \Rightarrow (AA^{-1})^T=I^T$

$\Rightarrow (A^{-1})^T A^T=I \Rightarrow (A^{-1})^T=(A^T)^{-1}$

$\Rightarrow (A^{-1})^T=A^{-1} \quad (\because A=A^T)$

Hence the inverse of a symmetrix matrix is symmetric.

54. Let $A=\begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix}$, then the given equation is

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix}=\begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x-u & 2y-v & 2z-w \\ x & y & z \\ -3x+4u & -3y+4v & -3z+4w \end{bmatrix}=\begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

Equating the corresponding elements, we get the values of x, y, z, u, v, and w as 1, –2, –5, 3, 4, and 0.

Hence $\quad A=\begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix}$

55. $A^2=\begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} a & b \\ b & a \end{bmatrix}=\begin{bmatrix} a^2+b^2 & 2ab \\ 2ab & a^2+b^2 \end{bmatrix}$

But $A^2=\begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$

$\therefore \quad \alpha=a^2+b^2,\ \beta=2ab$

56. If A is a non-singular matrix of order n, then

$$\text{adj}\,(\text{adj}A)=|A|^{n-2}A$$

Here, $n = 3$.

$\therefore \quad \text{adj}(\text{adj}A)=|A|A$.

57. $AB = A$ and $BA = B$

Now, $AB=A \Rightarrow (AB)A=AA=A^2$

$\Rightarrow A(BA)=A^2 \Rightarrow AB=A^2 \quad (\because BA=B)$

$\Rightarrow A=A^2 \quad (\because AB=A)$

$\Rightarrow A$ is idempotent. Similarly, B is idempotent.

58. $\because PQ=I \Rightarrow P^{-1}=Q$

Now, the system in matrix notation is $PX = B$.

$\therefore \quad X=P^{-1}B=QB$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix}=\frac{1}{9}\begin{bmatrix} 2 & 2 & 1 \\ 13 & -5 & m \\ -8 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$$

$\Rightarrow \quad y=\frac{1}{9}(13-5+5m)$

$\Rightarrow \quad -27=8+5m \qquad$ (given $y=-3$)

$\therefore \quad m=-7$

59. We have, $A^2=A$

$\therefore \quad (I+A)^2=(I+A)(I+A)=I+2A+A^2=I+3A$

and $(I+A)^3=(I+A)^2(I+A)$

$=(I+3A)(I+A) \quad [\because (I+A)^2=I+3A]$

$=I+4A+3A^2=I+7A \quad [\because A^2=A]$

Thus, we have

$(I+A)^2=I+3A$ and $(I+A)^3=I+7A$

$\Rightarrow \quad (I+A)^2=I+(2^2-1)A$

and $\quad (I+A)^3=I+(2^3-1)A$

Hence, $(I+A)^n=I+(2^n-1)A$

$\therefore \quad (I+A)^n=I+\lambda A \Rightarrow \lambda=2^n-1$

60. A is skew symmetric.

$\Rightarrow \quad A^T=-A \Rightarrow (A^T)^n=(-A)n$

$\Rightarrow \quad (A^n)^T=(-1)^n A^n=\begin{cases} A^n \text{ if } n \text{ is even} \\ -A^n \text{ if } n \text{ is odd} \end{cases}$

Hence the option (d).

61. $\begin{bmatrix} \omega & \omega^2 \\ 1 & \omega \\ \omega^2 & 1 \end{bmatrix}\times\begin{bmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \end{bmatrix}$

$$=\begin{bmatrix} \omega+\omega^3 & \omega^2+\omega^4 & \omega^3+\omega^2 \\ 1+\omega^2 & \omega+\omega^3 & \omega^2+\omega \\ \omega^2+\omega & \omega^3+\omega^2 & \omega^4+1 \end{bmatrix}$$

$$=\begin{bmatrix} -\omega^2 & -1 & -\omega \\ -\omega & -\omega^2 & -1 \\ -1 & -\omega & -\omega^2 \end{bmatrix}$$

Now, $-\begin{bmatrix} \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \\ 1 & \omega & \omega^2 \end{bmatrix}\begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

62. The given system of equations has a non-trivial solution.

$$\therefore \quad \begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix}=0$$

$\Rightarrow\ 7\sin 3\theta + 14\cos 2\theta - 14 = 0$

$\Rightarrow\ 3\sin\theta - 4\sin^3\theta + 2 - 4\sin^2\theta - 2 = 0$

$\Rightarrow\ \sin\theta(4\sin^2\theta + 4\sin\theta - 3) = 0$

$\Rightarrow\ \sin\theta(2\sin\theta + 3)(2\sin\theta - 1) = 0$

$\Rightarrow\ \sin\theta = 0$ or, $\sin\theta = (1/2)$

$\Rightarrow\ \theta = n\pi$ or $\theta = n\pi + (-1)^n \dfrac{\pi}{6}$

63. As A is symmetric $\Rightarrow A^T = A$
Also, as A is skew-symmetric

$\Rightarrow\ A^T = -A$

$\therefore\ A = -A \Rightarrow 2A = 0$

$\Rightarrow\ A = 0$

$\Rightarrow\ A$ is null matrix.

64. Now, $A + B = \begin{bmatrix} 3 & 4 \\ 4 & 4 \end{bmatrix} + \begin{bmatrix} -2 & -2 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$

$\Rightarrow\ |A+B| = \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = 2 - 4 = -2$

Also, adj $(A+B) = \begin{bmatrix} 2 & -2 \\ -2 & 1 \end{bmatrix}$

$\therefore\ (A+B)^{-1} = -\dfrac{1}{2}\begin{bmatrix} 2 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1/2 \end{bmatrix}$

Hence $(A + B)^{-1}$ exists. It is not a skew-symmetric matrix.

Now, $A^{-1} = \dfrac{1}{4}\begin{bmatrix} 4 & -4 \\ -2 & 3 \end{bmatrix}$ and $B^{-1} = \dfrac{1}{4}\begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}$

$\therefore\ A^{-1} + B^{-1} = \begin{bmatrix} 1 & -1 \\ -1/2 & 3/4 \end{bmatrix} + \begin{bmatrix} -1/2 & 1/2 \\ 0 & -1/2 \end{bmatrix}$

$= \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/4 \end{bmatrix}$

$\therefore\ (A+B)^{-1} \neq A^{-1} + B^{-1}$

$\therefore$ Option (d) is correct.

65. We have trace of $A = 2a$.
Clearly, trace of A will be divisible by p iff $a = 0$.

$\therefore\ a \in \{1, 2, ..., (p-1)\}$

We have $\det(A) = a^2 - bc$

$\therefore\ p\,|\det(A) \Rightarrow p\,|a^2 - bc$

$\Rightarrow\ a^2 - bc = \lambda p$ for some integer λ

For each value of $a \in \{1, 2, \ldots, (p-1)\}$, there are $(p-1)$ ordered pairs (b, c) for bc such that $a^2 - bc$ is divisible by p.
Hence, there are $(p-1)^2$ matrices in T_p such that the trace of A is not divisible by p but the determinant of A is divisible by p.

HINTS AND SOLUTIONS—EXERCISE SET 2

1. As $|AB| = |A||B|$

$\because\ AB = 0 \Rightarrow |AB| = 0$

$\Rightarrow\ |A| \cdot |B| = 0$

i.e. either $|A| = 0$ or $|B| = 0$.

2. $A\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$

Multiplying both sides by $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}^{-1}$, we get

$A = \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}^{-1}$,

but $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}^{-1} = \dfrac{\begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix}}{10 - 12} = \dfrac{1}{2}\begin{bmatrix} -5 & 3 \\ 4 & -2 \end{bmatrix}$

$\therefore\ A = \dfrac{1}{2}\begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}\begin{bmatrix} -5 & 3 \\ 4 & -2 \end{bmatrix} = \dfrac{1}{2}\begin{bmatrix} -16 & 8 \\ -38 & 24 \end{bmatrix}$

$= \begin{bmatrix} -8 & 4 \\ -19 & 12 \end{bmatrix}$

3. Since $A(\text{adj}A) = |A| I_n$

$\therefore\ |A||\text{adj}A| = |A|^n$

$\Rightarrow\ |\text{adj}A| = |A|^{n-1}$ $\quad [\because |A| \neq 0]$

4. $|A| = 1(16 - 9) - 3(4 - 3) + 3(3 - 4) = 1$

$\because\ A^{-1} = \dfrac{1}{|A|}\text{adj}A$, hence $A^{-1} = \text{adj}A$

5. $\det(A) = \begin{vmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{vmatrix} = 0$

$\Rightarrow\ -(x-3)^2 x = 0 \Rightarrow x = 0, 3$

6. We have $A = \begin{bmatrix} 1 & \log_b a \\ \log_a b & 1 \end{bmatrix}$

$\therefore\ |A| = 1 - \log_a b \log_a a = 1 - 1 = 0$

7. $AA^{-1} = I \Rightarrow |AA^{-1}| = |I|$

$\Rightarrow\ |A||A^{-1}| = 1 \Rightarrow |A^{-1}| = \dfrac{1}{|A|}$ $\quad [\because |A| \neq 0]$

8. Here $|A| = 4$

$\therefore \quad |5A| = 5^3 \times 4 = 500$

(Note that 5 is taken common from all the three columns or rows)

9. $A^2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \ldots, A^n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$

$= \begin{bmatrix} n & 0 \\ n & n \end{bmatrix} - \begin{bmatrix} n-1 & 0 \\ 0 & n-1 \end{bmatrix}$

$= nA - (n-1)I$

10. Given that $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 1 & 1 \end{bmatrix}$

$\therefore \quad AB = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2-6+1 & 1-4+1 \\ 4+3+3 & 2+2+3 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 10 & 7 \end{bmatrix}$

$\Rightarrow (AB)^T = \begin{bmatrix} -3 & 10 \\ -2 & 7 \end{bmatrix}$

11. A is orthogonal $\Rightarrow \quad AA^T = A^T A = I$

$\Rightarrow |AA^T| = |I| \quad \Rightarrow \quad |A||A^T| = 1$

$\Rightarrow |A||A| = 1 \quad [\because |A| = |A^T|] \quad \Rightarrow \quad |A| = \pm 1$

12. Now, $BB^T = (I-A)^{-1}(I+A)(I+A)^T[(I-A)^{-1}]^T$

$= (I-A)^{-1}(I+A)(I-A)(I+A)^{-1}$

$= (I-A)^{-1}(I-A)(I+A)(I+A)^{-1}$

$= I \cdot I$

$= I$

$\Rightarrow B$ is an orthogonal matrix.

13. As $P^3 = P(I-P) \qquad [\because P^2 = I - P]$

$= PI - P^2 = PI - (I-P)$

$= P - I + P = 2P - I$

Now, $P^4 = P.P^3$

$\Rightarrow \quad P^4 = P(2P - I) \quad \Rightarrow \quad P^4 = 2P^2 - P$

$\Rightarrow \quad P^4 = 2I - 2P - P \quad \Rightarrow \quad P^4 = 2I - 3P$

and $\quad P^5 = P(2I - 3P)$

$\Rightarrow \quad P^5 = 2P - 3(I-P) \quad \Rightarrow \quad P^5 = 5P - 3I$

Also, $P^6 = P(5P - 3I)$

$\Rightarrow \quad P^6 = 5P^2 - 3P \quad \Rightarrow \quad P^6 = 5(I-P) - 3P$

$\Rightarrow \quad P^6 = 5I - 8P$

So, $n = 6$

Alternate solution

As $P^n = 5I - 8P \quad \Rightarrow \quad P^n = 5(I-P) - 3P$

$\Rightarrow P^n = P(5P - 3I) \qquad (\because P^2 = I - P)$

$\Rightarrow P^n = P(2P - 3P^2) \quad \Rightarrow \quad P^n = P^2(2I - 3P)$

$\Rightarrow P^n = P^2[2(I-P) - P]$

$\Rightarrow P^n = P^2[2P^2 - P] \quad \Rightarrow \quad P^n = P^3[2P - I]$

$\Rightarrow P^n = P^4[I - P] \quad \Rightarrow \quad P^n = P^4 \cdot P^2 = P^6$

$\Rightarrow n = 6$

14. L.H.S. of option (a) is a matrix while R.H.S. is a determinant which is a number. Hence (a) is not true.

15. $A^{-1}(A^2 - 3A + 2I) = A^{-1} \cdot 0 = 0$

$\Rightarrow \quad 2A^{-1} = 3I - A \quad \Rightarrow \quad A^{-1} = \frac{3I - A}{2}$

16. Given matrix A is a square matrix and $AA^T = I = A^T A$

$\Rightarrow |AA^T| = |I| = |A^T A| \quad \Rightarrow \quad |A||A^T| = 1 = |A^T||A|$

$\Rightarrow |A|^2 = 1 \qquad [\because |A||A^T| = |A|^2]$

$\Rightarrow |A| = \pm 1$

17. $|A| = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} = a^3$

$\Rightarrow \qquad |A| \cdot |\text{adj} A| = ||A| \cdot I||A|^3$

$\Rightarrow \qquad |\text{adj} A| = |A|^2 = (a^3)^2 = a^6$

18. $\qquad x - 2y + z = -4 \qquad$ (i)

$2x - y + 2z = 2 \qquad$ (ii)

$x + y + \lambda z = 4 \qquad$ (iii)

$\Rightarrow \qquad x - 2y + z = -4$

From Eqs. (i) and (ii), $3y = 10$

From Eqs. (iii) and (i), $3y + (\lambda - 1)z = 8$

$\Rightarrow \qquad (\lambda - 1)z = -2$

If $\lambda = 1, 0 \cdot z = -2$, no solution.

If $\lambda \neq 1$, we have unique solution.

19. Given: $M = [a_{uv}]_{n \times n} = [\sin(\theta_u - \theta_v) + i\cos(\theta_u - \theta_v)]$

$\Rightarrow \qquad \overline{M} = [\sin(\theta_u - \theta_v) - i\cos(\theta_u - \theta_v)]$

$\Rightarrow \quad (\overline{M})^T = [\sin(\theta_v - \theta_u) - i\cos(\theta_v - \theta_u)]$

$= [-\sin(\theta_u - \theta_v) - i\cos(\theta_u - \theta_v)]$

$= -[\sin(\theta_u - \theta_v) + i\cos(\theta_u - \theta_v)]$

$= -M$

$\Rightarrow \quad M = -(\overline{M})^T$

20. The two-rowed unit matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$\therefore \quad \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} \alpha^2 + \beta\gamma & 0 \\ 0 & \beta\gamma + \alpha^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \quad \alpha^2 + \beta\gamma = 1 \quad \Rightarrow \quad 1 - \alpha^2 - \beta\gamma = 0$

21. $\begin{vmatrix} 1 & -k & -1 \\ k & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix} = 0$

$\Rightarrow \quad \begin{vmatrix} 1 & -k & -1 \\ k-1 & k-1 & 0 \\ 0 & k+1 & 0 \end{vmatrix} = 0$

[using $R_2 - R_1 \to R_2, R_3 - R_1 \to R_3$]

$\Rightarrow \quad k^2 - 1 = 0 \quad \Rightarrow \quad k = 1, -1$

22. We have $(AA^T)^T = (A^T)^T A^T = AA^T$

$\therefore \quad AA^T$ is a symmetric matrix.

23. $|A^n| = |A \cdot A \ldots A|$

$= |A| \cdot |A| \ldots |A| = |A|^n = 2^n$

24. $A = \begin{bmatrix} \frac{\omega}{i} & \frac{\omega^2}{i} \\ -\frac{\omega^2}{i} & \frac{-\omega}{i} \end{bmatrix} = \frac{\omega}{i}\begin{bmatrix} 1 & \omega \\ -\omega & -1 \end{bmatrix}$

$\therefore \quad A^2 = -\omega^2 \begin{bmatrix} 1-\omega^2 & 0 \\ 0 & 1-\omega^2 \end{bmatrix}$

$= -\begin{bmatrix} -\omega^2 + \omega^4 & 0 \\ 0 & -\omega^2 + \omega^4 \end{bmatrix}$

$= \begin{bmatrix} -\omega^2 + \omega & 0 \\ 0 & -\omega^2 + \omega \end{bmatrix}$

$\because \quad f(x) = x^2 + 2$

$\therefore \quad f(A) = A^2 + 2I$

$= \begin{bmatrix} -\omega^2 + \omega & 0 \\ 0 & -\omega^2 + \omega \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$= [-\omega^2 + \omega + 2]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (3 + 2\omega)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= (2 + i\sqrt{3})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

25. Since $\begin{vmatrix} 3 & 3 \\ 5 & 5 \end{vmatrix} = 0$,

$\therefore \quad \begin{bmatrix} 3 & 3 \\ 5 & 5 \end{bmatrix}$ is not invertible.

26. Given that $A^{-1} = \lambda(\text{adj} A)$

On comparing with $A^{-1} = \frac{1}{|A|}$ adj A, we get

$\lambda = \frac{1}{|A|}$

Now, $|A| = \begin{vmatrix} 0 & 3 \\ 2 & 0 \end{vmatrix} = 0 - 6 = -6$

$\Rightarrow \quad \lambda = -\frac{1}{6}$

27. $A + B$ is defined.

$\therefore \quad A$ and B are of same order, i.e. AB is defined.

$\therefore \quad$ The number of colums in A equals the number of rows in B. Hence A and B are square matrices of same order.

28. $\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$

$= \begin{bmatrix} \cos\theta\cos\phi\cos(\theta \sim \phi) & \cos\theta\sin\phi\cos(\theta \sim \phi) \\ \cos\phi\sin\theta\cos(\theta \sim \phi) & \sin\theta\sin\phi\cos(\theta \sim \phi) \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O \quad \left[\text{If } \theta - \phi = (2n+1)\frac{\pi}{2}, n \in Z\right]$

29. $A^2 = AA = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$

$= \begin{bmatrix} i^2 + 0 & 0 + 0 \\ 0 + 0 & 0 + i^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$A^4 = A^2 A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1+0 & 0-0 \\ -0-0 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Hence, $A^{4n} = (A^4)^n = (I)^n = I$

30. Given $A^2 = 2A - I$

Now, $A^3 = A^2 \cdot A = 2A^2 - IA$

$= 2A^2 - A = 2(2A - I) - A$

$= 3A - 2I = 3A - (3-1)I \ldots$

$A^n = nA - (n-1)I$ for $n \geq 2$

31. $|A+B|$ cannot be expressed in terms of $|A|$ and $|B|$. Hence the given equation gives no inference.

32. $|P^T AP| = |A|$

$\Rightarrow \quad |P^T||A||P| - |A| = 0$

$\Rightarrow \quad |A|(|P|^2 - 1) = 0, \ |A| \neq 0 \quad [\because \ |P^T| = |P|]$

$\Rightarrow \quad |P|^2 - 1 = 0 \quad \Rightarrow \quad |P| = \pm 1$

33. The given matrix A is singular if

$$|A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & \lambda \end{vmatrix} = 0$$

$\Rightarrow \ 8(7\lambda - 16) + 6(-6\lambda + 8) + 2(24 - 14) = 0$

$\Rightarrow \ 56\lambda - 128 - 36\lambda + 48 + 20 = 0$

$\Rightarrow \ 20\lambda = 60$

$\Rightarrow \ \lambda = 3$

34. By expanding the determinant, we get

$= 1(1 - \omega^{3n}) - \omega^n(\omega^{2n} - \omega^{2n}) + \omega^{2n}(\omega^{4n} - \omega^n)$

$= (1-1) - 0 + \omega^{2n}(\omega^n - \omega^n) = 0$

35. $\because \ |kA| = k^n|A|$, where k is a scalar and A is a matrix of the order $n \times n$. Here, A is of the order 3×3 and $k = 3$.

$\therefore \ |3A| = 3^3|A| = 27|A|$

36. Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

$$\therefore \quad |A| = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = 0\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} - 1\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$= -(1-9) + 2(1-6) = 8 - 10 = -2$

and $\text{adj}\, A = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix}^T = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

Hence, $A^{-1} = \dfrac{1}{|A|}(\text{adj}\, A)$

$$= -\frac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

37. Since $A(\text{adj}\, A) = |A|I$,

$\therefore \ |A||\text{adj}\, A| = |A||I| = |A|^3 \quad [\because \ A$ is a 3×3 matrix$]$

$\Rightarrow \ |\text{adj}\, A| = |A|^2$

38. $A = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$

$\therefore \quad |A| = \cos^2 2\theta + \sin^2 2\theta = 1$

and $\quad \text{adj}\, A = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$

$\therefore \quad A^{-1} = \dfrac{1}{1}\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$

$= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$

39. Since α, β, γ are the roots of $x^3 + px + q = 0$,

$\therefore \quad \alpha + \beta + \gamma = 0$

Using $R_1 \to R_1 + R_2 + R_3$, we get

$$\begin{bmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{bmatrix} = \begin{bmatrix} \alpha+\beta+\gamma & \alpha+\beta+\gamma & \alpha+\beta+\gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{bmatrix}$$

$= 0$

$[\because \quad \alpha + \beta + \gamma = 0]$

40. Given that $F(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow \quad F(-\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore \ F(\alpha)F(-\alpha)$

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha\sin\alpha - \sin\alpha\cos\alpha & 0 \\ \sin\alpha\cos\alpha - \sin\alpha\cos\alpha & \sin^2\alpha + \cos^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\Rightarrow \quad [F(\alpha)]^{-1} = F(-\alpha)$

41. Given that p is a non-singular matrix such that

$$1+p+p^2+\cdots+p^n=O$$

$\Rightarrow$ $(1+p)(1+p+p^2+\cdots+p^n)=O$

$\Rightarrow$ $1-p^{n+1}=O$

$\Rightarrow$ $p^{n+1}=1$

$\Rightarrow$ $p^n\times p^1=1$

$\Rightarrow$ $p^n=1/p$

$\therefore$ $p^{-1}=p^n$

42. $\because$ $A^TA=\frac{1}{3}\begin{bmatrix}1&-2&2\\-2&1&2\\-2&-2&-1\end{bmatrix}\cdot\frac{1}{3}\begin{bmatrix}1&-2&-2\\-2&1&-2\\2&2&-1\end{bmatrix}$

$$=\frac{1}{9}\begin{bmatrix}9&0&0\\0&9&0\\0&0&9\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I$$

$\therefore$ A is orthogonal.

Also, if A and B are orthogonal, then AB is orthogonal.

43. Let $A=\begin{bmatrix}0&-c&b\\c&0&a\\-b&-a&0\end{bmatrix}$, then $A=-A^T$

$\therefore$ $\det(A)=\det(-A^T)=-\det(A^T)=-\det A$

$\therefore$ $\det A=0$

$\det A^T=\det(-A^T)$ is not true

$\because$ $\det(-A^T)=(-1)^3\det(A^T)=-\det A^T$

44. $\because$ $|A|=\begin{bmatrix}3&4\\3&5\end{bmatrix}=15-12=3\neq 0$

$\therefore$ A is a non-singular matrix.

$\therefore$ A^{-1} exists.

45. Let $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$, $B=\begin{bmatrix}x&y\\z&u\end{bmatrix}$

Also, $AB=BA$ (given)

$\Rightarrow$ $\begin{bmatrix}ax+bz&ay+bu\\cx+dz&cy+du\end{bmatrix}=\begin{bmatrix}ax+cy&bx+dy\\az+cu&bz+du\end{bmatrix}$

On comparing, we get

$$ax+bz=ax+cy$$

$\Rightarrow$ $bz=cy$

$\Rightarrow$ $\frac{z}{c}=\frac{y}{b}=\lambda$ (say)

$\Rightarrow$ $y=b\lambda,\quad z=c\lambda$ (i)

and $ay+bu=bx+dy$

$\Rightarrow$ $ab\lambda+bu=bx+bd\lambda$ [From Eq. (i)]

$\Rightarrow$ $a\lambda+u=x+d\lambda=k$ (say)

For $\lambda=0$; $y=0$, $z=0$, $u=k$, $x=k$

Then, $B=\begin{bmatrix}k&0\\0&k\end{bmatrix}$ = scalar matrix

Also, if $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ and $B=\begin{bmatrix}k&0\\0&k\end{bmatrix}$

Then, $AB=BA=\begin{bmatrix}ak&bk\\ck&dk\end{bmatrix}=kA$

46. If $A=[a_{ij}]_{n\times n}$ is a square matrix such that $a_{ij}=0$ for $i\neq j$; then A is called a diagonal matrix. Thus, the given statement is true and $A=\begin{bmatrix}3&0&0\\0&4&0\\0&0&7\end{bmatrix}$ is a diagonal matrix.

47. Appling $C_1\to xC_1$ and $C_1\to C_1+(C_2-C_3)$, we get the given matrix as

$$\frac{1}{x}\begin{vmatrix}0&b&ax+b\\0&c&bx+c\\ax^2+2bx+c&bx+c&0\end{vmatrix}$$

$$=\frac{(ax^2+2bx+c)}{x}[b^2x+bc-acx-bc]$$

$$=(b^2-ac)(ax^2+2bx+c)$$

$$=(+\text{ve})\,(-\text{ve})$$

$$<0$$

48. Let A be the first term and R be the common ratio of G.P.

$\therefore$ $l=t_P=AR^{p-1}$

$\Rightarrow$ $\log l=\log A+(p-1)\log R$

Similarly, $\log m=\log A+(q-1)\log R$

and $\log n=\log A+(r-1)\log R$

$\therefore$ $\begin{vmatrix}\log l&p&1\\\log m&q&1\\\log n&r&1\end{vmatrix}=\begin{vmatrix}\log A+(p-1)\log R&p&1\\\log A+(q-1)\log R&q&1\\\log A+(r-1)\log R&r&1\end{vmatrix}$

$$=\underset{C_1\propto C_3}{\begin{vmatrix}\log A-\log R&p&1\\\log A-\log R&q&1\\\log A-\log R&r&1\end{vmatrix}}+\underset{C_1\propto C_2}{\begin{vmatrix}p\log R&p&1\\q\log R&q&1\\r\log R&r&1\end{vmatrix}}$$

$$=0+0=0$$

49. Here $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & 2b \\ 3a & 4b \end{bmatrix}$

$$BA = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a & 2a \\ 3b & 4a \end{bmatrix}$$

Hence, $AB = BA$ only when $a = b$. It means infinitely B's such that $AB = BA$.

50. $A^2 = \begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} a^2+b^2 & 2ab \\ 2ab & a^2+b^2 \end{bmatrix}$

But $A^2 = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$

$\Rightarrow$ $\alpha = a^2 + b^2$ and $\beta = 2ab$

51. For a non-trivial solution, the determinant of the coefficients of various terms vanishes

i.e. $\begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{vmatrix} = 0$

$\Rightarrow$ $(3bc - 4bc) - 2a(c-b) + a(4c-3b)$

$\Rightarrow$ $\dfrac{2ac}{a+c} = b$

$\Rightarrow$ a, b, c are in H.P.

52. As vectors $(1, a, a^2)$, $(1, b, b^2)$ and $(1, c, c^2)$ are non-coplanar.

$\therefore$ $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \neq 0$

Now $\begin{vmatrix} a & a^2 & a^3+1 \\ b & b^2 & b^3+1 \\ c & c^2 & c^3+1 \end{vmatrix} = 0$

$\Rightarrow$ $(1+abc)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$

$\Rightarrow$ $(1+abc) = 0$ $\Rightarrow$ $abc = -1$

53. (i) $|A| = 1$, $\therefore$ A^{-1} does not exist is a wrong statement.

(ii) $(-1)I = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \neq A \Rightarrow (b)$ is false

(iii) A is clearly a non-zero matrix.

$\therefore$ (c) is false

Now, we are left with option (d) only.

54. Here, $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$

$\therefore$ $|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix}$

$= 1\cdot(1+3) + 1\cdot(2+3) + 1\cdot(2-1)$

$= 4 + 5 + 1 = 10$

$\Rightarrow$ adj $A = \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}^T = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$

$\Rightarrow$ $B = A^{-1} = \dfrac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$

$\Rightarrow$ $(10)B = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix}$

Hence, $\alpha = 5$

55. $A^2 - A + I = 0 \Rightarrow I = A - A \cdot A$

$IA^{-1} = AA^{-1} - A(AA^{-1})$ or $A^{-1} = I - A$

56. $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$\therefore$ $A^2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $A^3 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$, so $A^n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$

and

$$nA - (n-1)I = \begin{pmatrix} n & 0 \\ n & n \end{pmatrix} - \begin{pmatrix} n-1 & 0 \\ 0 & n-1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} = A^n$$

57. Applying $C_2 \to C_2 + C_3 + C_1$, we get

$$f(x) = 1 + 2x + x(a^2+b^2+c^2)\begin{vmatrix} 1+a^2x & 1 & (1+c^2)x \\ (1+a^2)x & 1 & (1+c^2)x \\ (1+a^2)x & 1 & 1+c^2x \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$ and using $a^2 + b^2 + c^2 = -2$, we have

$$f(x) = (1+2x-2x)\begin{vmatrix} 1-x & 0 & 0 \\ 0 & 0 & x-1 \\ (1+a^2)x & 1 & 1+c^2x \end{vmatrix} = (1-x)^2$$

$= x^2 - 2x + 1$

$\therefore$ Degree of $f(x)$ is 2.

58. Let $\Delta = \begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = \alpha^3 - 3\alpha + 2 = 0$

$\Rightarrow \quad (\alpha - 1)(\alpha^2 + \alpha - 2) = 0$

$\Rightarrow \quad (\alpha - 1)^2(\alpha + 2) = 0$

$\Rightarrow \alpha = 1$ or -2

$\alpha = 1 \Rightarrow x + y + z = 0$ gives an infinite number of solutions. Hence, $\alpha = -2$ has no solution.

59. Given: $A^2 - B^2 = (A + B)(A - B)$

$\Rightarrow \quad 0 = BA - AB \Rightarrow BA = AB$

60. $|A^2| = 25 \Rightarrow |A| = \pm 5$

$\Rightarrow 625\alpha^2 = \pm 5$

$\Rightarrow \alpha = \frac{1}{5}$

61. $D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{bmatrix}$

(Appling $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$)

$D = \begin{bmatrix} 1 & 0 & 0 \\ 1 & x & 0 \\ 1 & 0 & y \end{bmatrix} = 1(xy - 0) = xy$

Hence, D is divisible by both x and y.

62. Given equations have a non-trivial solution if the determinant of the coefficient matrix is zero.

$\therefore \quad \begin{bmatrix} 1 & -c & -b \\ -c & 1 & -a \\ -b & -a & 1 \end{bmatrix} = 0$

$\Rightarrow \quad 1 - a^2 - b^2 - c^2 - 2abc = 0$

$\Rightarrow \quad a^2 + b^2 + c^2 + 2abc = 1$

63. Given: A is a square matrix with all entries as integers.

(a) Now $|A| \neq 1, -1$ means it may be zero. Also if $|A| = 0$ then A^{-1} does not exist.

$\therefore$ Choice (a) is false.

(b) If $|A| = 1, -1$, then A^{-1} certainly exists but A is a square matrix with all integral entries so all cofactors are integers. So adj A matrix has all integral entries.

$\therefore \quad A^{-1} = \frac{1}{|A|}(\text{adj}A) = \pm(\text{adj}A)$

$\therefore$ Choice (b) is correct.

(c) $|A| = 1, -1$, $\therefore A^{-1}$ must exist but given that A^{-1} does no texist which is false. $\therefore$ Choice (c) is incorrect.

(d) $|A| = 1, -1$, it is true that A^{-1} exists but we have to follow that A has all integral entries but choice (d) says that it is not necessary that entries are integers which means adj A may or may not be with integral entries. Hence, the choice (d) is false.

64. Given that $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$ and $A^2 = B$

$\Rightarrow \quad \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} \alpha^2 & 0 \\ \alpha + 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$

$\Rightarrow \quad \alpha^2 = 1, \alpha + 1 = 5$

$\Rightarrow \quad \alpha = \pm 1, \alpha = 4$

$\therefore$ There is no common value. Hence there is no real value of α for which $A^2 = B$.

65. Since A is a 3×4 matrix,

$\Rightarrow A^T$ is a 4×3 matrix

Also A^TB is defined

$\Rightarrow (B)_{4\times3}$

Also, BA^T is defined $\Rightarrow B_{3\times4}$

$\therefore$ Order of B is 3×4.

66. Given: $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$

$A \cdot A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$

$= \begin{bmatrix} a^2b^2 - a^2b^2 & ab^3 - ab^3 \\ -a^3b + a^3b & -a^2b^2 + a^2b^2 \end{bmatrix}$

$= 0$

$\therefore \quad A^2 = 0$

$\therefore \quad A$ is a nilpotent matrix of order 2.

67. $A + B = I \Rightarrow 2A + 2B = 2I$ (i)

and $2A - 2B = I$ (ii)

Adding (i) and (ii),

$\Rightarrow \quad A = \frac{3}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

68. $\begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} -x & 14x & 7x \\ 0 & 1 & 0 \\ x & -4x & -2x \end{bmatrix}$

$= \begin{bmatrix} 5x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10x - 2 & 5x \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (given)

$\therefore \quad 5x = 1, 10x - 2 = 0, \quad \therefore \quad x = (1/5)$

69. $(A^{-1}BA)^2 = (A^{-1}BA)(A^{-1}BA)$

$= A^{-1}B(AA^{-1})BA \;\; = A^{-1}BIBA = A^{-1}B^2A$

$\Rightarrow \; (A^{-1}BA)^3 = (A^{-1}B^2A)(A^{-1}BA)$

$= A^{-1}B^2(AA^{-1})BA$

$= A^{-1}B^2IBA \;\; = A^{-1}B^3A$, and so on

$\Rightarrow \; (A^{-1}BA)^n = A^{-1}B^nA$

70. As $A\,\text{adj}\,A = |A|I$

$\Rightarrow \; |A\,\text{adj}\,A| = |A|^n$ [If A is of order $n \times n$]

$\Rightarrow \; |A||\text{adj}\,A| = |A|^n \;\; \Rightarrow \;\; |\text{adj}\,A| = |A|^{n-1}$

As A is singular, $|A| = 0$

$\Rightarrow \; |\text{adj}\,A| = 0$

Hence, adj A is singular.

71. $H^2 = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}\begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix} = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix}$

If $H^k = \begin{bmatrix} \omega^k & 0 \\ 0 & \omega^k \end{bmatrix}$, then $H^{k+1} = \begin{bmatrix} \omega^{k+1} & 0 \\ 0 & \omega^{k+1} \end{bmatrix}$

So by mathematical induction,

$H^{70} = \begin{bmatrix} \omega^{70} & 0 \\ 0 & \omega^{70} \end{bmatrix} = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix} = H$

72. $x - ky + z = 0$

$kx + 3y - kz = 0$

$3x + y - z = 0$

These equations will have non-trivial solution if

$$\begin{vmatrix} 1 & -k & 1 \\ k & 3 & -k \\ 3 & 1 & -1 \end{vmatrix} = 0$$

$1(-3+k) + k(-k+3k) + 1(k-9) = 0$

$\Rightarrow \; k - 3 + 2k^2 + k - 9 = 0 \;\Rightarrow\; 2k^2 + 2k - 12 = 0$

$\Rightarrow \; k^2 + k - 6 = 0 \qquad \Rightarrow \; k = -2, k = 2$

So the equations will have only trivial solution when $k \in \mathbb{R} - \{2, -3\}$.

73. A = $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$. Let $u_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$; $u_2 = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$

$Au_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \;\Rightarrow\; \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a \\ 2a+b \\ 3a+2b+c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \;\Rightarrow\; a = 1;\; b = -2;\; c = 1$

$\Rightarrow \; u_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $Au_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \;\Rightarrow\; \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ 2x+y \\ 3x+2y+z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \;\Rightarrow\; x = 0;\; y = 1;\; z = -2$

$\Rightarrow \; u_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

$\therefore \;\; u_1 + u_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

74. $P^3 = Q^3$ (i)

$Q^2P = P^2Q$ (ii)

Adding Eq. (i) and (ii)

$P^3 + Q^2P = Q^3 + P^2Q$

$P(P^2 + Q^2) = Q(Q^2 + P^2)$

$(P - Q)(P^2 + Q^2) = 0$

$\Rightarrow$ det $(P^2 + Q^2) = 0$ $(\because \;\; P - Q \neq 0)$

MULTIPLE CHOICE TYPE QUESTIONS—LEVEL 1

1. The number of matrices having 12 elements is

(a) 3 (b) 1

(c) 6 (d) none of these

2. If a matrix A is symmetric as well as skew-symmetric, then A is

(a) a diagonal matrix (b) a null matrix

(c) a unit matrix (d) none of these

3. If $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 4 & 0 \\ 0 & 2 & -1 \\ 1 & -3 & 2 \end{bmatrix}$, then

(a) $AB = \begin{bmatrix} -5 & 8 & 0 \\ 0 & 4 & -2 \\ 3 & -9 & 6 \end{bmatrix}$ (b) $AB = \begin{bmatrix} -2 & -1 & 4 \end{bmatrix}$

(c) $AB = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ (d) AB does not exist

4. If A and B are two square matrices such that $AB = 0$, then

(a) det $A = 0$ or det $B = 0$ (b) det $B = 0$

(c) $B = A^{-1}$ (d) det $A = 0$

5. If $A=\begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}$ and $B=\begin{bmatrix}a_1&a_2&a_3\\b_1&b_2&b_3\\c_1&c_2&c_3\end{bmatrix}$, then AB is equal to

(a) B (b) $3B$
(c) B^3 (d) $A + B$

6. If $A=\begin{bmatrix}1&2\\2&1\end{bmatrix}$, then adj A = ?

(a) $\begin{bmatrix}1&-2\\-2&1\end{bmatrix}$ (b) $\begin{bmatrix}2&1\\1&1\end{bmatrix}$
(c) $\begin{bmatrix}1&-2\\-2&-1\end{bmatrix}$ (d) $\begin{bmatrix}-1&2\\2&-1\end{bmatrix}$

7. If $I=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $J=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$ and $B=\begin{bmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{bmatrix}$, then B = ?

(a) $I\cos\theta+J\sin\theta$ (b) $I\cos\theta-J\sin\theta$
(c) $I\sin\theta+J\cos\theta$ (d) $-I\cos\theta+J\sin\theta$

8. If $A=\begin{bmatrix}0&0&0\\1&0&0\\0&1&0\end{bmatrix}$, then

(a) $A^2 = A$ (b) $A^2 = 0$
(c) $A^2 = I$ (d) $A^3 = 0$

9. If A is a skew-symmetric matrix, then the trace of A is

(a) 1 (b) –1
(c) 0 (d) none of these

10. If $A=\begin{bmatrix}\cos\theta&-\sin\theta&0\\\sin\theta&\cos\theta&0\\0&0&1\end{bmatrix}$, then adj A = ?

(a) A^T (b) I
(c) 0 (d) A^2

11. If $A=\begin{vmatrix}0&0&0\\0&0&0\\0&1&0\end{vmatrix}$, then A is

(a) an invertible matrix (b) an idempotent matrix
(c) a nilpotent matrix (d) none of these

12. If A is any square matrix, then

(a) $A + A^T$ is skew-symmetric
(b) $A - A^T$ is symmetric
(c) AA^T is symmetric
(d) none of these

13. For any 2 × 2 matrix A, if A(adj A) = $\begin{bmatrix}10&0\\0&10\end{bmatrix}$, then $|A|$ is equal to

(a) 20 (b) 100
(c) 10 (d) 0

14. If A and B are square matrices of order 3 such that $|A|=-1$ and $|B|=3$, then $|3AB|$ is equal to

(a) –9 (b) –81
(c) –27 (d) 81

15. If A and B are square matrices of order 2, then $(A+B)^2$ is equal to

(a) $A^2+2AB+B^2$ (b) $A^2+AB+BA+B^2$
(c) $A^2+2BA+B^2$ (d) none of these

16. If A is a square matrix such that $A^3 = I$ then A^{-1} is equal to

(a) I (b) A
(c) A^2 (d) none of these

17. If B is a non-singular matrix and A is a square matrix, then $\det(B^{-1}AB)$ is equal to

(a) $\det(A^{-1})$ (b) $\det(B^{-1})$
(c) $\det(A)$ (d) $\det(B)$

18. Matrix $\begin{bmatrix}a&b&(a\alpha-b)\\b&c&(b\alpha-c)\\2&1&0\end{bmatrix}$ is non-invertible if

(a) $\alpha=\frac{1}{2}$ (b) a, b, c are in A.P.
(c) a, b, c are in G.P. (d) none of these

19. Let A be a skew-symmetric matrix of order n. Then

(a) $|A|=0$ if n is even (b) $|A|=0$ if n is odd
(c) $|A|=0$ for all $n\in N$ (d) none of these

20. Let A and B be 3 × 3 matrices. Then $AB = 0$ implies

(a) $A = 0$ and $B = 0$
(b) $|A|=0$ and $|B|=0$
(c) either $|A|=0$ or $|B|=0$
(d) $A = 0$ or $B = 0$

21. If A, B are two $n \times n$ non-singular matrices, then

(a) AB is non-singular (b) AB is singular
(c) $(AB)^{-1}=A^{-1}B^{-1}$ (d) $(AB)^{-1}$ does not exist

22. If $A=\begin{bmatrix}1&2\\3&-5\end{bmatrix}$ and $B=\begin{bmatrix}1&0\\0&2\end{bmatrix}$ and X be matrices such that $A = BX$, then X is equal to

(a) $\frac{1}{2}\begin{bmatrix}2&4\\3&-5\end{bmatrix}$ (b) $\frac{1}{2}\begin{bmatrix}-2&4\\3&5\end{bmatrix}$
(c) $\begin{bmatrix}2&4\\3&-5\end{bmatrix}$ (d) none of these

23. Each diagonal element of a skew-symmetric matrix is

(a) zero (b) positive
(c) non-real (d) negative

24. If the matrix $A=\begin{bmatrix} y+a & b & c \\ a & y+b & c \\ a & b & y+c \end{bmatrix}$ has rank 3, then

(a) $y \neq (a+b+c)$
(b) $y \neq 1$
(c) $y = 0$
(d) $y \neq -(a+b+c)$ and $y \neq 0$

25. If $A=\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, $n \in N$ then A^{4n} is

(a) $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$

26. If A is a square matrix such that $A^2 = I$, then $A^{-1} = ?$

(a) $2A$ (b) A
(c) 0 (d) $A + I$

27. Let a, b, c, d, u, v be integers. If the system of equations $ax + by = u$, and $cx + dy = v$ has a unique solution in integers, then

(a) $ad - bc = 1$
(b) $ad - bc = -1$
(c) $ad - bc \neq 0$
(d) $ad - bc$ need not be equal to ± 1

28. If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ then A^{-1} is equal to

(a) $\begin{bmatrix} 7 & -3 & -3 \\ 0 & 1 & 0 \\ -1 & 0 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 7 & -3 & -3 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ (d) none of these

29. If A is a non-zero column matrix of order $m \times 1$ and B is a non-zero row matrix of order $1 \times n$, then the rank of AB is equal to

(a) n (b) m
(c) 1 (d) none of these

30. The matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is

(a) an idempotent matrix
(b) an involutory matrix
(c) a nilpotent matrix
(d) none of these

MULTIPLE CHOICE TYPE QUESTIONS—LEVEL 2

1. If A = dig (2, – 1, 3) and B = dig (– 1, 3, 2), then A^2B =

(a) dig (5, 4, 11) (b) dig (– 4, 3, 18)
(c) dig (3, 1, 8) (d) B

2. If $A=\begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}$ and $B=\begin{bmatrix} 4 & 1 \\ 7 & 2 \end{bmatrix}$, then B^TA^T is

(a) a null matrix
(b) an identity matrix
(c) a scalar matrix, but not an identity matrix
(d) such that $Tr(B^T A^T) = 4$

3. Which of the following relations is true for $A=\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $B=\begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}$?

(a) $(A+B)^2 = A^2 + 2AB + B^2$
(b) $(A-B)^2 = A^2 - 2AB + B^2$
(c) $AB = BA$
(d) none of these

4. The trace of a skew-symmetric matrix is always equal to

(a) $\sum a_{ij}$ (b) $\sum a_{ii}$
(c) zero (d) none of these

5. Which of the following is incorrect?

(a) $A^2 - B^2 = (A+B)(A-B)$
(b) $(A^T)^T = A$
(c) $(AB)^n = A^n B^n$, where A and B commute
(d) $(A-I)(I+A) = 0 \Leftrightarrow A^2 = I$

6. If $X=\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then value of X^n is

(a) $\begin{bmatrix} 3n & -4n \\ n & -n \end{bmatrix}$ (b) $\begin{bmatrix} 2+n & 5-n \\ n & -n \end{bmatrix}$

(c) $\begin{bmatrix} 3^n & (-4)^n \\ 1^n & (-1)^n \end{bmatrix}$ (d) none of these

7. If the matrix AB is a zero matrix, then

(a) $A = 0$ or $B = 0$
(b) $A = 0$ and $B = 0$
(c) it is not necessary that either $A = 0$ or $B = 0$
(d) all the above statements are wrong

8. If A^T is the transpose of a square matrix A, then
(a) $|A| \neq |A^T|$
(b) $|A| = |A^T|$
(c) $|A| + |A^T| = 0$
(d) $|A| = |A^T|$ only when A is symmetric

9. If $AB = 0$ for the matrices $A = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$ and $B = \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$ then $\phi - \theta$ is
(a) an odd multiple of $\pi/2$
(b) an odd multiple of π
(c) an even multiple of $\pi/2$
(d) 0

10. If A is a skew-symmetric matrix and n is an even positive integer, then A^n is
(a) a symmetric matrix
(b) a skew-symmetric matrix
(c) a diagonal matrix
(d) none of these

11. If I_n is the identity matrix of order n, then $(I_n)^{-1}$
(a) does not exist (b) equals I_n
(c) equals O (d) none of these

12. If A and B are symmetric matrices, then ABA is a
(a) symmetric matrix (b) skew-symmetric matrix
(c) diagonal matrix (d) scalar matrix

13. If A is a non-singular matrix and A^T denotes the transpose of A, then
(a) $|A| \neq |A^T|$ (b) $|A \cdot A^T| \neq |A|^2$
(c) $|A^T \cdot A| \neq |A^T|^2$ (d) $|A| + |A^T| \neq 0$

14. Given $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$; $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. If $A - \lambda I$ is a singular matrix, then
(a) $\lambda \in \phi$ (b) $\lambda^2 - 3\lambda - 4 = 0$
(c) $\lambda^2 + 3\lambda + 4 = 0$ (d) $\lambda^2 - 3\lambda - 6 = 0$

15. If for a matrix A of order 2, $A^2 + I = 0$, where I is the identity matrix, then A equals
(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$
(c) $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

16. From the matrix equation $AB = AC$, we conclude that $B = C$ provided
(a) A is singular (b) A is non-singular
(c) A is symmetric (d) A is square

17. If A is a square matrix of order 3, then the true statement is (where I is the unit matrix)
(a) $\det(-A) = -\det A$
(b) $\det A = 0$
(c) $\det(A + I) = 1 + \det A$
(d) $\det 2A = 2\det A$

18. If A is a square matrix of order $n \times n$ and k is a scalar, then adj (kA) is equal to
(a) k adj A (b) k^n adj A
(c) k^{n-1} adj A (d) k^{n+1} adj A

19. If A and B are symmetric matrices of order $n\,(A \neq B)$, then
(a) $A + B$ is a skew-symmetric matrix
(b) $A + B$ is a symmetric matrix
(c) $A + B$ is a diagonal matrix
(d) $A + B$ is a zero matrix

20. If $F(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $G(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$, then $[F(\alpha)G(\beta)]^{-1} =$
(a) $F(\alpha) - G(\beta)$ (b) $-F(\alpha) - G(\beta)$
(c) $[F(\alpha)]^{-1}[G(\beta)]^{-1}$ (d) $[G(\beta)]^{-1}[F(\alpha)]^{-1}$

21. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfies the equation $x^2 - (a + b)x + k = 0$, then
(a) $k = bc$ (b) $k = ad$
(c) $k = a^2 + b^2 + c^2 + d^2$ (d) $k = ad - bc$

22. The singularity of matrix
$$\begin{bmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x & \sin px & \sin(p+d)x \end{bmatrix}$$
depends on which of the following parameters?
(a) a (b) p
(c) x (d) none of these

23. If $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ and $B = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$, then $AB =$
(a) A^3 (b) B^2
(c) 0 (d) I

24. Which of the following is true for matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

(a) $A + 4I$ is a symmetric matrix

(b) $A - B$ is a diagonal matrix for any value of α if $B = \begin{bmatrix} \alpha & -1 \\ 2 & 5 \end{bmatrix}$

(c) $A - 4I$ is a skew-symmetric matrix

(d) none of these

25. The value of a for which the system of equations

$$a^3x + (a+1)^3 y + (a+2)^3 z = 0$$
$$ax + (a+1)y + (a+2)z = 0$$
$$x + y + z = 0$$

has a non-zero solution is

(a) -1 (b) 0

(c) 1 (d) none of these

26. The value of k for which the set of equations

$$3x + ky - 2z = 0,\ x + ky + 3z = 0$$

and $$2x + 3y - 4z = 0$$

has a non-trivial solution over the set of rationals is

(a) $\frac{33}{2}$ (b) $\frac{31}{2}$

(c) 16 (d) 15

27. If $AB = A$ and $BA = B$, then B^2 is equal to

(a) B (b) A

(c) 1 (d) 0

28. The values of x for which the matrix

$$\begin{bmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{bmatrix}$$

is non-singular are

(a) $R - \{0\}$ (b) $R - \{-a(a+b+c)\}$

(c) $R - \{0, -(a+b+c)\}$ (d) none of these

29. Which of the following is a nilpotent matrix?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

30. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, then $19A^{-1}$ is equal to

(a) A^T (b) $2A$

(c) $\frac{1}{2}A$ (d) A

ANSWERS

Level 1

1. (c)	**2.** (b)	**3.** (d)	**4.** (a)	**5.** (b)	**6.** (a)	**7.** (a)	**8.** (d)	**9.** (c)	**10.** (a)
11. (c)	**12.** (c)	**13.** (c)	**14.** (b)	**15.** (b)	**16.** (c)	**17.** (c)	**18.** (a)	**19.** (b)	**20.** (c)
21. (a)	**22.** (a)	**23.** (a)	**24.** (d)	**25.** (c)	**26.** (b)	**27.** (c)	**28.** (b)	**29.** (c)	**30.** (c)

Level 2

1. (b)	**2.** (b)	**3.** (d)	**4.** (c)	**5.** (a)	**6.** (d)	**7.** (c)	**8.** (b)	**9.** (a)	**10.** (a)
11. (b)	**12.** (a)	**13.** (d)	**14.** (b)	**15.** (b)	**16.** (b)	**17.** (a)	**18.** (c)	**19.** (b)	**20.** (d)
21. (d)	**22.** (c)	**23.** (c)	**24.** (b)	**25.** (a)	**26.** (a)	**27.** (a)	**28.** (c)	**29.** (c)	**30.** (d)

To be continued ...

Prepared by –

SANJAY BHATTACHARYA, (Asst. Teacher)

SUKUMAR MANDAL, (Asst. Teacher)

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