# ST. LAWRENCE HIGH SCHOOL

A JESUIT CHRISTIAN MINORITY INSTITUTION

FOR GOD AND COUNTRY

## STUDY MATERIAL-24

**SUBJECT – MATHEMATICS**

**Pre-Test**

**Chapter: Integration** **Class: XII**

**Topic: Some standard Forms** **Date: 07.07.2020**

# -:Some standard Forms:-

# (Part 4)

**5.** $\int \frac{x^2}{x^4+1}dx, \int \frac{1}{x^4+1}dx, \int \frac{x^2+a^2}{x^4+kx^2+1}dx, \int \frac{x^2-a^2}{x^4+kx^2+1}dx, (k \in R)$

**Rule for this form:**

**(a)** To evaluate these types of integrals divide the numerator and denominator by $x^2$.

**(b)** Put $x+\frac{1}{x}=t$ or $x-\frac{1}{x}=t$ and $x+\frac{a^2}{x}=t$ or $x-\frac{a^2}{x}=t$ as required.

**Similar form is:** $\int \sqrt{\tan x}\, dx, \int \sqrt{\cot x}\, dx, \int \frac{dx}{\sin^4 x+\cos^4 x},$

$$\int \frac{dx}{\sin^6 x+\cos^6 x}, \int \frac{(\pm \sin x \pm \cos x)}{a+b\sin x}dx$$

**Example 1.** Evaluate $\int \frac{5}{x^4+1}dx$.

**Solution:**

$$I=\int \frac{5}{x^4+1}dx=\frac{5}{2}\left(\int \frac{(x^2+1)-(x^2-1)}{x^4+1}dx\right)$$

$$=\frac{5}{2}\left(\int \frac{\left(1+\frac{1}{x^2}\right)}{x^2+\frac{1}{x^2}}dx-\int \frac{\left(1-\frac{1}{x^2}\right)}{x^2+\frac{1}{x^2}}dx\right)$$

$$=\frac{5}{2}\left(\int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x-\frac{1}{x}\right)^2+2}dx-\int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)^2-2}dx\right)=\frac{5}{2}(I_1-I_2)$$

$$I_1=\int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x-\frac{1}{x}\right)^2+2}dx$$

For $I_1$, we write

$$x-\frac{1}{x}=t \Rightarrow \left(1+\frac{1}{x^2}\right)dx=dt$$

$$I_1=\int \frac{1}{t^2+2}dt$$

$$I_1=\frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}+c=\frac{1}{\sqrt{2}}\tan^{-1}\frac{\left(x-\frac{1}{x}\right)}{\sqrt{2}}+c_1$$

For $I_2$, we write

$$x+\frac{1}{x}=t \Rightarrow \left(1-\frac{1}{x^2}\right)dx=dt$$

$$I_2=\int \frac{1}{t^2-2}dt=\frac{1}{2\sqrt{2}}\ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right|+c=\frac{1}{2\sqrt{2}}\ln\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right|+c_2$$

Combining the two integrals, we get

$$I=\frac{5}{2}\left(\frac{1}{\sqrt{2}}\tan^{-1}\frac{\left(x-\frac{1}{x}\right)}{\sqrt{2}}-\frac{1}{2\sqrt{2}}\ln\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right|\right)+k$$

**Example 2.** Evaluate $\int\sqrt{\tan x}\,dx$.

**Solution:** Put $\tan x=t^2$. Then

$$\sec^2 x\,dx=2tdt\Rightarrow dx=\frac{2t}{1+t^4}dt$$

$$I=\int\sqrt{\tan x}\,dx=\int\frac{2t\cdot t}{1+t^4}dt=\int\frac{t^2+1}{1+t^4}dt+\int\frac{t^2-1}{1+t^4}dt$$

$$I=\int\frac{\left(1+\frac{1}{t^2}\right)}{\left(t-\frac{1}{t}\right)^2+2}dx+\int\frac{\left(1-\frac{1}{t^2}\right)}{\left(t-\frac{1}{t}\right)^2-2}dx$$

$$I=I_1+I_2$$

For $I_1$, we write $\left(t-\frac{1}{t}\right)=z\Rightarrow\left(1+\frac{1}{t^2}\right)dx=dz$

$$I_1=\int\frac{1}{z^2+2}dt$$

$$I_1=\frac{1}{\sqrt{2}}\tan^{-1}\frac{z}{\sqrt{2}}+c=\frac{1}{\sqrt{2}}\tan^{-1}\frac{\left(t-\frac{1}{t}\right)}{\sqrt{2}}+c$$

$$I_1=\frac{1}{\sqrt{2}}\tan^{-1}\frac{\left(\sqrt{\tan x}-\frac{1}{\sqrt{\tan x}}\right)}{\sqrt{2}}+c$$

For $I_2$, we write $t+\frac{1}{t}=z \Rightarrow \left(1-\frac{1}{t^2}\right)dx = dz$

$$I_2 = \int \frac{1}{z^2-2}dt = \frac{1}{2\sqrt{2}}\ln\left|\frac{z-\sqrt{2}}{z+\sqrt{2}}\right| + c = \frac{1}{2\sqrt{2}}\ln\left|\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right| + c_1$$

$$I_2 = \frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{\tan x}+\frac{1}{\sqrt{\tan x}}-\sqrt{2}}{\sqrt{\tan x}+\frac{1}{\sqrt{\tan x}}+\sqrt{2}}\right| + c_2$$

Combining the two integrals, we get

$$I = \left(\frac{1}{\sqrt{2}}\tan^{-1}\frac{\left(\sqrt{\tan x}-\frac{1}{\sqrt{\tan x}}\right)}{\sqrt{2}} + \frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{\tan x}+\frac{1}{\sqrt{\tan x}}-\sqrt{2}}{\sqrt{\tan x}+\frac{1}{\sqrt{\tan x}}+\sqrt{2}}\right|\right) + k$$

**6. Substitution for some irrational functions:**

**(a)** $\int f(x,(ax+b)^{1/n})dx$, put $(ax+b)=t^n$

**(b)** $\int \frac{dx}{(px+q)\sqrt{(ax+b)}}$, put $(ax+b)=t^2$

**(c)** $\int \frac{dx}{(px+q)\sqrt{(ax^2+bx+c)}}$, put $(px+q)=\frac{1}{t}$

**(d)** $\int \frac{dx}{(px^2+q)\sqrt{(ax^2+b)}}$, first put $x=\frac{1}{t}$ and then $(a+bt^2)=z^2$

**(e)** $\int \frac{(ax^2+bx+c)dx}{(dx+e)\sqrt{(fx^2+gx+h)}}$, here, we write

$$ax^2+bx+c = A_1(dx+e)(2fx+g)+B_1(dx+e)+C_1$$

where $A_1$, $B_1$ and $C_1$ are constants which can be obtained by comparing the coefficient of like terms on both sides.

**Example 3.** Evaluate $\int \frac{x\,dx}{(x-3)\sqrt{x+1}}$.

**Solution:** Put $x+1=t^2$. Then $dx=2tdt$, we get

$$I=\int \frac{2t(t^2-1)}{(t^2-4)t}\,dt$$

$$=2\int\left(1+\frac{3}{(t^2-4)}\right)dt=2t+\frac{3}{2}\ln\left|\frac{t-2}{t+2}\right|+c$$

$$=2\sqrt{x+1}+\frac{3}{2}\ln\left|\frac{\sqrt{x+1}-2}{\sqrt{x+1}+2}\right|+c$$

**Example 4.** Evaluate $\int \frac{dx}{(x-3)\sqrt{x+1}}$.

**Solution:** Put $x+1=t^2$. Then $dx=2tdt$, we get

$$I=\int \frac{2t}{(t^2-4)t}\,dt=2\int \frac{1}{(t^2-4)}\,dt=2\frac{1}{4}\ln\left|\frac{t-2}{t+2}\right|+c$$

$$=\frac{1}{2}\ln\left|\frac{\sqrt{x+1}-2}{\sqrt{x+1}+2}\right|+c$$

**Example 5.** Evaluate $\int \frac{dx}{(x+1)\sqrt{(x^2-x+1)}}$.

**Solution:**

$$I=\int \frac{dx}{(x+1)\sqrt{(x^2-x+1)}}$$

Put $x+1=\frac{1}{t}$. Then $dx=-\frac{1}{t^2}dt$, we get

$$I=-\int \frac{1}{t^2.\frac{1}{t^2}\sqrt{(1-t)^2-t(1-t)+t^2}}\,dt=-\int \frac{1}{\sqrt{3t^2-3t+1}}\,dt$$

$$I=-\frac{1}{\sqrt{3}}\int\frac{1}{\sqrt{\left(t-\frac{1}{2}\right)^2+\frac{1}{12}}}dt=-\frac{1}{\sqrt{3}}\ln\left|\left(t-\frac{1}{2}\right)+\sqrt{\left(t-\frac{1}{2}\right)^2+\frac{1}{12}}\right|+c$$

$$I=-\frac{1}{\sqrt{3}}\ln\left|\left(\frac{1-x}{2(1+x)}\right)+\sqrt{\left(\frac{1-x}{2(1+x)}\right)^2+\frac{1}{12}}\right|+c$$

**Example 6.** Evaluate $\int\frac{(2x^2+3x+2)dx}{(x+1)\sqrt{(x^2-x+1)}}$

**Solution:**

$$I=\int\frac{(2x^2+3x+2)dx}{(x+1)\sqrt{(x^2-x+1)}}$$

$$(2x^2+3x+2)=a(x+1)\frac{d}{dx}(x^2-x+1)+b(x+1)+c$$

$$a=1,b=2,c=1$$

$$I=\int\frac{(x+1)(2x-1)+2(x+1)+1}{(x+1)\sqrt{(x^2-x+1)}}dx$$

$$I=\int\frac{(2x-1)}{\sqrt{(x^2-x+1)}}dx+2\int\frac{1}{\sqrt{(x^2-x+1)}}dx+\int\frac{1}{(x+1)\sqrt{(x^2-x+1)}}dx$$

$$I=I_1+2I_2+I_3$$

$$I_1=\int\frac{(2x-1)}{\sqrt{(x^2-x+1)}}dx=2\sqrt{(x^2-x+1)}+c_1$$

$$I_2=\int\frac{1}{\sqrt{(x^2-x+1)}}dx=\int\frac{1}{\sqrt{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}}$$

$$=\ln\left|\left(x-\frac{1}{2}\right)+\sqrt{(x^2-x+1)}\right|+c_2$$

$$I_3=\int\frac{1}{(x+1)\sqrt{(x^2-x+1)}}dx$$

$$= -\frac{1}{\sqrt{3}}\ln\left|\left(\frac{1-x}{2(1+x)}\right)+\sqrt{\left(\frac{1-x}{2(1+x)}\right)^2+\frac{1}{12}}\right|+c_3$$

$$I = 2\sqrt{(x^2-x+1)}+2\ln\left|\left(x-\frac{1}{2}\right)+\sqrt{(x^2-x+1)}\right|-$$

$$\frac{1}{\sqrt{3}}\ln\left|\left(\frac{1-x}{2(1+x)}\right)+\sqrt{\left(\frac{1-x}{2(1+x)}\right)^2+\frac{1}{12}}\right|+c$$

**Example 7.** Evaluate $\int \frac{dx}{(x^2+2)\sqrt{x^2+1}}$.

**Solution:**

$$I = \int \frac{dx}{(x^2+2)\sqrt{x^2+1}}$$

Put $x = \frac{1}{t}$. Then $dx = -\frac{1}{t^2}dt$, we get

$$I = -\int \frac{dt}{t^2\left(\frac{1}{t^2}+2\right)\sqrt{\frac{1}{t^2}+1}} = -\int \frac{tdt}{(1+2t^2)\sqrt{t^2+1}}$$

Put $(1+t^2)=z^2$. Then $tdt = zdz$, we get

$$I = -\int \frac{z\,dz}{\left[1+2(z^2-1)\right]z} = -\int \frac{dz}{(2z^2-1)}$$

$$I = -\frac{1}{2}\int \frac{dz}{\left(z^2-\frac{1}{2}\right)} = -\frac{1}{2\sqrt{2}}\ln\left|\frac{z-\frac{1}{\sqrt{2}}}{z+\frac{1}{\sqrt{2}}}\right|+c$$

$$I = -\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}z-1}{\sqrt{2}z+1}\right|+c = -\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2(1+t^2)}-1}{\sqrt{2(1+t^2)}+1}\right|+c$$

$$I = -\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2(1+x^2)}-x}{\sqrt{2(1+x^2)}+x}\right|+c$$


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