ST. LAWRENCE HIGH SCHOOL

A JESUIT CHRISTIAN MINORITY INSTITUTION

FOR GOD AND COUNTRY

STUDY MATERIAL-25

SUBJECT – MATHEMATICS

Pre-Test

Chapter: Integration **Class: XII**

Topic: Some standard Forms **Date: 08.07.2020**

-:Some standard Forms:-

(Part 5)

7. Substitution for trigonometric functions:

(a) $\int \frac{dx}{(a+b\cos x)}, \int \frac{dx}{(a+b\sin x)}$

Use $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}, \sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and put $\tan\frac{x}{2} = t.$

Illustration Evaluate $\int \frac{dx}{(2+\cos x)}$.

Solution:

$$I=\int \frac{dx}{(2+\cos x)}=\int \frac{dx}{\left(2+\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}=\int \frac{\sec^2\frac{x}{2}}{3+\tan^2\frac{x}{2}}dx$$

Put $\tan\frac{x}{2}=t$. Then $\frac{1}{2}\sec^2\frac{x}{2}dx=dt$.

$$I=2\int \frac{dt}{(3+t^2)}=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)+c=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt{3}}\right)+c$$

Illustration Evaluate $\int \frac{dx}{(5+4\cos x)}$.

Solution:

$$I=\int \frac{dx}{(5+4\cos x)}$$

$$I=\int \frac{\left(1+\tan^2\frac{x}{2}\right)dx}{\left(5\left(1+\tan^2\frac{x}{2}\right)+4\left(1-\tan^2\frac{x}{2}\right)\right)}=\int \frac{\sec^2\frac{x}{2}}{9-\tan^2\frac{x}{2}}dx$$

Put $\tan\frac{x}{2}=t$. Then $\frac{1}{2}\sec^2\frac{x}{2}dx=dt$.

(b) $\int \frac{dx}{(a\sin x + b\cos x + c)}, \int \frac{dx}{(a\cos x + b\sin x)}$

Use $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}, \sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and put $\tan\frac{x}{2} = t$

Or $a = r\cos\alpha$ and $b = r\sin\alpha \Rightarrow r = \sqrt{a^2+b^2}$ and $\alpha = \tan^{-1}\frac{b}{a}$.

Illustration Evaluate $\int \frac{dx}{(\sin x + \cos x + 2)}$.

Solution:

$$I = \int \frac{dx}{(\sin x + \cos x + 2)} = \int \frac{\left(1+\tan^2\frac{x}{2}\right)dx}{\left(2\left(1+\tan^2\frac{x}{2}\right) + \left(1-\tan^2\frac{x}{2}\right) + 2\tan\frac{x}{2}\right)}$$

$$I = \int \frac{\sec^2\frac{x}{2}dx}{\left(\tan^2\frac{x}{2} + 2\tan\frac{x}{2} + 3\right)}$$

Put $\tan\frac{x}{2} = t$. Then $\frac{1}{2}\sec^2\frac{x}{2}dx = dt$.

$$I = 2\int \frac{dt}{(t^2+2t+3)} = 2\int \frac{dt}{\left[(t+1)^2+2\right]} = \frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{t+1}{\sqrt{2}}\right) + c$$

$$\Rightarrow I = \sqrt{2}\tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{\sqrt{2}}\right) + c$$

Illustration Evaluate $\int \frac{dx}{(4\sin x + 3\cos x)}$.

Solution:

$$I = \int \frac{dx}{(4\sin x + 3\cos x)}$$

$$3\cos x + 4\sin x = 5\left(\frac{3}{5}\cos x + \frac{4}{5}\sin x\right) = 5\cos(x-\alpha), \tan\alpha = \frac{4}{3}$$

$$I = \frac{1}{5}\int \frac{dx}{\cos(x-\alpha)} = \frac{1}{5}\int \sec(x-\alpha)dx = \frac{1}{5}[\sec(x-\alpha) + \tan(x-\alpha)] + c$$

Illustration Evaluate $\int \frac{dx}{(1-\sin x - \cos x)}$.

Solution:

$$I = \int \frac{dx}{(1-\sin x - \cos x)} = \int \frac{dx}{\left(1 - \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} - \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}$$

$$I = \int \frac{\left(1+\tan^2\frac{x}{2}\right)dx}{\left(1+\tan^2\frac{x}{2} - 2\tan\frac{x}{2} - 1 + \tan^2\frac{x}{2}\right)}$$

$$I = \int \frac{\sec^2\frac{x}{2}dx}{\left(2\tan^2\frac{x}{2} - 2\tan\frac{x}{2}\right)}$$

Put $\tan\frac{x}{2} = t$. Then $\frac{1}{2}\sec^2\frac{x}{2}dx = dt$. Therefore,

$$I = \int \frac{dt}{(t^2-t)} = \int \frac{dt}{t(t-1)} = \int \left(\frac{1}{t-1} - \frac{1}{t}\right)dt$$

$$I = \ln(t-1) - \ln t + c = \ln\left|\frac{t-1}{t}\right| + c$$

$$\Rightarrow I = \ln\left|\frac{\tan\frac{x}{2}-1}{\tan\frac{x}{2}}\right| + c$$

(c) $\int \frac{dx}{(a+b\cos^2 x)}, \int \frac{dx}{(a+b\sin^2 x)},$

$\int \frac{dx}{(a\sin^2 x + b\cos^2 x + c)}, \int \frac{dx}{(a\cos^2 x + b\sin^2 x)}, \int \frac{dx}{(a\cos x + b\sin x)^2}$

Divide the numerator and the denominator by $\sin^2 x$ or $\cos^2 x$ and put $\tan x = t$.

Illustration Evaluate $\int \frac{dx}{(1+2\cos^2 x)}$.

Solution:

$$I = \int \frac{dx}{(1+2\cos^2 x)} = \int \frac{\sec^2 x\, dx}{(\sec^2 x + 2)} = \int \frac{\sec^2 x\, dx}{(\tan^2 x + 3)}$$

(Dividing the numerator and the denominator by $\cos^2 x$)

Put $\tan x = t$. Then $\sec^2 x\, dx = dt$.

$$I = \int \frac{dt}{(t^2+3)}$$

$$I = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + c = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\tan x}{\sqrt{3}}\right) + c$$

Illustration Evaluate $\int \frac{dx}{(3\cos x + \sin x)^2}$.

Solution:

$$I = \int \frac{dx}{(3\cos x + \sin x)^2}$$

(Dividing the numerator and the denominator by $\cos^2 x$)

$$I = \int \frac{dx}{(3\cos x + \sin x)^2} = \int \frac{\sec^2 x\, dx}{(3+\tan x)^2}$$

Put $\tan x = t$. Then $\sec^2 x\, dx = dt$.

$$I=\int\frac{dt}{(3+t)^2}=-\frac{1}{(3+t)}+c$$

$$I=-\frac{1}{(3+\tan x)}+c$$

Illustration Evaluate $\int\frac{\cos x}{\cos 3x}dx$.

Solution:

$$I=\int\frac{\cos x}{\cos 3x}dx=\int\frac{\cos x}{(4\cos^3 x-3\cos x)}dx=\int\frac{1}{(4\cos^2 x-3)}dx$$

(Dividing the numerator and the denominator by $\cos^2 x$)

$$I=\int\frac{\sec^2 x}{(4-3(1+\tan^2 x))}dx=\int\frac{\sec^2 x}{(1-3\tan^2 x)}dx$$

Put $\tan x=t$. Then $\sec^2 x\,dx=dt$.

$$I=\int\frac{1}{(1-3t^2)}dt=\frac{1}{3}\int\frac{1}{\left(\frac{1}{3}-t^2\right)}dt$$

$$I=\frac{1}{2\sqrt{3}}\ln\left|\frac{1+\sqrt{3}t}{1-\sqrt{3}t}\right|+c=\frac{1}{2\sqrt{3}}\ln\left|\frac{1+\sqrt{3}\tan x}{1-\sqrt{3}\tan x}\right|+c$$

Illustration Evaluate $\int\frac{dx}{(4\sin^2 x+5\cos^2 x+4\sin x\cos x)}$

Solution:

$$I=\int\frac{dx}{(4\sin^2 x+5\cos^2 x+4\sin x\cos x)}$$

(Dividing the numerator and the denominator by $\cos^2 x$)

$$I=\int\frac{dx}{(4\sin^2 x+5\cos^2 x+4\sin x\cos x)}=\int\frac{\sec^2 x dx}{(4\tan^2 x+5+4\tan x)}$$

$$I=\frac{1}{4}\int\frac{dx}{\left(\left(\tan x+\frac{1}{2}\right)^2+1\right)}$$

Put $\tan x+\frac{1}{2}=t$. Then $\sec^2 x\,dx=dt$.

$$I=\frac{1}{4}\int\frac{dx}{(t^2+1)}=\frac{1}{4}\tan^{-1}t+c=\frac{1}{4}\tan^{-1}\left(\tan x+\frac{1}{2}\right)+c$$

(d) $\int\frac{(a\cos x+b\sin x)}{(p\cos x+q\sin x)}dx$

$$(a\cos x+b\sin x)=l(p\cos x+q\sin x)+m\frac{d}{dx}(p\cos x+q\sin x)$$

Compare both side coefficients of $\sin x$ and $\cos x$, and calculate the value of l and m.

$$\int\frac{(a\cos x+b\sin x)}{(p\cos x+q\sin x)}dx$$

$$(a\cos x+b\sin x+c)=l(p\cos x+q\sin x+r)+m\frac{d}{dx}(p\cos x+q\sin x+r)+n$$

Compare both side coefficients of $\sin x$, $\cos x$ and constant term, and calculate the value of l, m and n.

Illustration Evaluate $\int\frac{(4\cos x+5\sin x)}{(2\cos x+3\sin x)}dx$.

Solution:

$$I=\int\frac{(4\cos x+5\sin x)}{(2\cos x+3\sin x)}dx$$

$$(4\cos x+5\sin x)=a(2\cos x+3\sin x)+b\frac{d}{dx}(2\cos x+3\sin x)$$
$$=a(2\cos x+3\sin x)+b(-2\sin x+3\cos x)$$

Comparing the coefficients of $\cos x$ and $\sin x$, we get

$$a=\frac{23}{13},\ b=\frac{2}{13}$$

$$I=\frac{23}{13}\int 1dx+\frac{2}{13}\int\frac{(3\cos x-2\sin x)}{(2\cos x+3\sin x)}dx$$
$$=\frac{23}{13}x+\frac{2}{13}\ln|2\cos x+3\sin x|+c$$

Illustration Evaluate $\int\frac{(\cos x-3\sin x+4)}{(\cos x+\sin x+2)}dx$.

Solution:

$$I=\int\frac{(\cos x-3\sin x+4)}{(\cos x+\sin x+2)}dx$$

$$(\cos x-3\sin x+4)=a(\cos x+\sin x+2)+b\frac{d}{dx}(\cos x+\sin x+2)+c$$
$$=a(\cos x+\sin x+2)+b(-\sin x+\cos x)+c$$

Comparing the coefficients of $\cos x$, $\sin x$ and constant, we get

$$a=-1,\ b=2,\ c=6$$

$$I=-\int 1dx+2\int\frac{(\cos x-\sin x)}{(\cos x+\sin x+2)}dx+6\int\frac{dx}{(\cos x+\sin x+2)}$$

For IIIrd integral,

$$6\int\frac{dx}{(\cos x+\sin x+2)}=6\int\frac{\left(1+\tan^2\frac{x}{2}\right)}{1-\tan^2\frac{x}{2}+2\tan\frac{x}{2}+2+2\tan^2\frac{x}{2}}dx$$

$$=6\int\frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2}+2\tan\frac{x}{2}+3}dx=6\int\frac{\sec^2\frac{x}{2}}{\left(1+\tan\frac{x}{2}\right)^2+2}dx$$

$$\Rightarrow I=-\int 1dx+2\int\frac{(\cos x-\sin x)}{(\cos x+\sin x+2)}dx+6\int\frac{\sec^2\frac{x}{2}}{\left(1+\tan\frac{x}{2}\right)^2+2}dx$$

$$I=-x+2\ln|(\cos x+\sin x+2)|+3\sqrt{2}\tan^{-1}\left(\frac{\left(1+\tan\frac{x}{2}\right)}{\sqrt{2}}\right)+c$$

Illustration Evaluate $\int \frac{dx}{1+\cot x}$.

Solution:

$$I=\int \frac{dx}{1+\cot x}=\int \frac{\sin x}{\sin x+\cos x}dx$$

$$\sin x=M\frac{d}{dx}(\sin x+\cos x)+N(\sin x+\cos x)$$
$$=M(-\sin x+\cos x)+N(\sin x+\cos x)$$

Comparing the coefficients of sin x and cos x, we have

$$-M+N=1,\ M+N=0$$

Solving these equations, we have $M=-\frac{1}{2}, N=\frac{1}{2}$.

$$\sin x=-\frac{1}{2}(-\sin x+\cos x)+\frac{1}{2}(\sin x+\cos x)$$

$$I=\int \frac{\sin x}{\sin x+\cos x}dx$$
$$=-\frac{1}{2}\int \frac{(-\sin x+\cos x)}{(\sin x+\cos x)}dx+\frac{1}{2}\int \frac{(\sin x+\cos x)}{(\sin x+\cos x)}dx$$
$$=-\frac{1}{2}\ln|(\sin x+\cos x)|+\frac{1}{2}x+c$$

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