ST. LAWRENCE HIGH SCHOOL

A JESUIT CHRISTIAN MINORITY INSTITUTION

FOR GOD AND COUNTRY

STUDY MATERIAL-14

SUBJECT – MATHEMATICS

Pre-test

Chapter: Second Order Derivative **Class: XII**

Topic: Second Order Derivative **Date: 19.06.2020**

-: Second Order Derivative :-

➢ Successive differentiation

The process of differentiating the same function again and again is called successive differentiation.

(i) The derivative of y with respect to x is called the first order derivative and is denoted by $\frac{dy}{dx}$ (or) y_1 (or) $f'(x)$

(ii) If $f'(x)$ is differentiable, then the derivative of $f'(x)$ with respect to x is called the second order derivative and is denoted by $\frac{d^2y}{dx^2}$ (or) y_2 (or) $f''(x)$

(iii) Further $\frac{d^n y}{dx^n}$ (or) y_n (or) $f^{(n)}(x)$ is called n^{th} order derivative of the function $y = f(x)$

Remarks

(i) If $y = f(x)$, then $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$.

(ii) If $x = f(t)$ and $y = g(t)$, then $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left\{\frac{g'(t)}{f'(t)}\right\}\cdot\frac{dt}{dx}$

Example 1.

Find the second order derivative of the following function:

$\boldsymbol{x^3 + \tan x}$

$$y = x^3 + \tan x$$

$$\frac{dy}{dx} = 3x^2 + \sec^2 x$$

$$\frac{d^2y}{dx^2} = 6x + 2\sec x.\sec x \tan x$$

$$= 6x + 2\sec^2 x.\tan x$$

Example 2.

If $x = \sin^{-1} t$ and $y = \log(1 - t^2)$, then $\frac{d^2y}{dx^2}$ at $t = 1/2$ is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{-2t}{1-t^2}}{\frac{1}{\sqrt{1-t^2}}} = \frac{-2t}{\sqrt{1-t^2}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) * \frac{dt}{dx} = \left(\frac{2\left(\sqrt{1-t^2}\right) - 2t\left(\frac{t}{\sqrt{1-t^2}}\right)}{1-t^2}\right)\left(1-t^2\right)$$

$$= \left(\frac{-2(1-t^2) - 2t^2}{1-t^2}\right)$$

$$\frac{d^2y}{dx^2}\Big|_{t=1/2} = \frac{-2}{3/4}$$

$$= \frac{-8}{3}$$

Example 3.

If $(a+bx)\,e^{y/x}=x$, then prove that $x^3\frac{d^2y}{dx^2}=\left(x\frac{dy}{dx}-y\right)^2$

Consider the given equation.

$(a+bx)\,e^{y/x}=x$

Take log on both the sides.

$$\ln(a+bx)+\frac{y}{x}\ln e=\ln x$$

$$\frac{y}{x}=\ln x-\ln(a+bx)$$

$$y=x\ln\left(\frac{x}{a+bx}\right)$$

So,

$$\frac{d}{dx}y=\frac{d}{dx}\left\{x\ln\left(\frac{x}{a+bx}\right)\right\}$$

$$\frac{dy}{dx}=\ln\left(\frac{x}{a+bx}\right)+x\left(\frac{a+bx}{x}\right)\left(\frac{a+bx-bx}{(a+bx)^2}\right)$$

$$\frac{dy}{dx}=\ln\left(\frac{x}{a+bx}\right)+\frac{a}{a+bx}$$

$$\frac{dy}{dx}=\ln\left(\frac{x}{a+bx}\right)+\frac{a}{a+bx}$$

$$\frac{dy}{dx}=\frac{y}{x}+\frac{a}{a+bx}$$

$$x\frac{dy}{dx}-y=\frac{ax}{a+bx}$$

Differentiating again with respect to x, we get

$$\frac{d^2y}{dx^2}=\frac{a^2}{x(a+bx)^2}$$

$$x^3\frac{d^2y}{dx^2}=\frac{a^2x^2}{(a+bx)^2}$$

From the above results,

$$x^3\frac{d^2y}{dx^2}=\left(x\frac{dy}{dx}-y\right)^2$$

Hence, $LHS=RHS$.

Example 4.

If $y=\left(\cot^{-1}x\right)^2$, then show that $\left(1+x^2\right)^2\frac{d^2y}{dx^2}+2x\left(1+x^2\right)\frac{dy}{dx}=2$.

$$y=(\cot^{-1}x)^2$$

$$y_1=\frac{dy}{dx}=\frac{d}{dx}(\cot^{-1}x)^2$$

$$y_1=\frac{-2\cot^{-1}x}{1+x^2} \quad \boxed{y_1=\frac{d}{dx}(\cot^{-1}x)^2}$$

$$(1+x^2)y_1=2\cot^{-1}x \quad \boxed{y_2=\frac{d^2(\cot^{-1}x)^2}{dx^2}}$$

squaring both the sides

$$(1+x^2)^2y_1^2=4(\cot^{-1}x)^2$$

differentiating w.r.t x

$$2(1+x^2)2x=y_1^2+(1+x^2)^2 2y,\forall y_2=4\times\frac{d}{dx}(\cot^{-1}x)^2$$

substituting (i)

$$2(1+x^2)2x.y_1^2+(1+x^2)^2 2y_1y_2=4.y_1$$

$$2y_1\left[2x(1+x^2)y_1+(1+x^2)y^2\right]=4y_1$$

$$\therefore\ (1+x^2)^2y_2+2x(1+x)^2y_1=2$$

$$\boxed{(1+x^2)^2.\frac{d^2y}{dx^2}+2x(1+x^2)\frac{dy}{dx}=2}$$

Hence proved.

Example 5.

If $x=\cos^n\theta, y=\sin^n\theta$ then $\frac{d^2y}{dx^2}=$

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{n\sin^{n-1}\theta\cos\theta}{-n\cos^{n-1}\theta\sin\theta}$$

$$=-\tan^{n-2}\theta$$

$$\frac{d^2y}{dx^2}=\frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx}$$

$$=\frac{-\left((n-2)\tan^{n-3}\sin\theta\right)}{-n\cos^{n-1}\theta\sin\theta}$$

$$=\frac{(n-2)}{n}\frac{\tan^{n-3}\theta}{\cos^{n+1}\theta\sin\theta}$$

Example 6.

If $\sqrt{x+y}+\sqrt{y-x}=c$ then $\frac{d^2y}{dx^2}$ equals

Given: $\sqrt{x+y}+\sqrt{y-x}=c$ ...(1)

$$\therefore \frac{1}{\sqrt{x+y}+\sqrt{y-x}}=\frac{1}{c}$$

Rationalising the denominator, we get

$$\Rightarrow \frac{\sqrt{x+y}-\sqrt{y-x}}{(x+y)-(y-x)}=\frac{1}{c}$$

$$\Rightarrow \frac{\sqrt{x+y}-\sqrt{y-x}}{2x}=\frac{1}{c}$$

$$\Rightarrow \sqrt{x+y}-\sqrt{y-x}=\frac{2x}{c} \quad ...(2)$$

By adding (1) and (2) we get

$$2\sqrt{y+x}=c+\frac{2x}{c}$$

$$\Rightarrow 4(y+x)=c^2+\frac{4x^2}{c^2}+4x$$

$$\therefore 4\left(\frac{dy}{dx}+1\right)=\frac{8x}{c^2}+4$$

$$\therefore 4\frac{dy}{dx}=\frac{8x}{c^2}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{2}{c^2}$$

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