ST. LAWRENCE HIGH SCHOOL

A JESUIT CHRISTIAN MINORITY INSTITUTION

STUDY MATERIAL-6

SUBJECT – MATHEMATICS

Pre-test

Chapter: MATRICES AND DETERMINANTS **Class: XII**

Topic: DETERMINANTS **Date: 14.05.2020**

PART 3

HINTS AND SOLUTIONS—EXERCISE SET 1

1. Taking x common from R_2 and $x(x-1)$ common from R_3, we get

$$f(x) = x^2(x-1)\begin{vmatrix} 1 & x & x+1 \\ 2 & x-1 & x+1 \\ 3 & x-2 & x+1 \end{vmatrix}$$

Applying $C_3 \to C_3 - C_2$, we get $f(x) = x^2(x-1)$

$$\begin{vmatrix} 1 & x & 1 \\ 2 & x-1 & 2 \\ 3 & x-2 & 3 \end{vmatrix} = 0$$

Thus, $f(500) = 0$

2. Since $f(\theta)=\begin{vmatrix}\cos^2\theta & \cos\theta\sin\theta & -\sin\theta\\ \cos\theta\sin\theta & \sin^2\theta & \cos\theta\\ \sin\theta & -\cos\theta & 0\end{vmatrix}$,

applying $C_1 \to C_1 - \sin\theta C_3$; $C_2 \to C_2 + \cos\theta C_3$

$$f(x)=\begin{vmatrix}1 & 0 & -\sin\theta\\ 0 & 1 & \cos\theta\\ \sin\theta & -\cos\theta & 0\end{vmatrix}$$

Applying $R_3 \to R_3 - \sin\theta R_1 + \cos\theta R_2$

$$f(\theta)=\begin{vmatrix}1 & 0 & -\sin\theta\\ 0 & 1 & \cos\theta\\ 0 & 0 & 1\end{vmatrix}$$

$\Rightarrow$ $f(\theta) = 1(1) = 1$ = constant function

$\therefore$ $f\left(\frac{\pi}{6}\right)=1$

3. Since α_1, α_2 and β_1, β_2 are the roots of $ax^2 + bx + c = 0$ and $px^2 + qx + r = 0$, respectively, therefore,

$$\alpha_1+\alpha_2=\frac{-b}{a},\ \alpha_1\alpha_2=\frac{c}{a}$$

and $\beta_1+\beta_2=\frac{-q}{p},\ \beta_1\beta_2=\frac{r}{p}$

Since the given system of equations has a non-trivial solution,

$\therefore$ $\begin{vmatrix}\alpha_1 & \alpha_2\\ \beta_1 & \beta_2\end{vmatrix}=0$ i.e. $\alpha_1\beta_2-\alpha_2\beta_1=0$

or $\frac{\alpha_1}{\beta_1}=\frac{\alpha_2}{\beta_2}=\frac{\alpha_1+\alpha_2}{\beta_1+\beta_2}=\sqrt{\frac{\alpha_1\alpha_2}{\beta_1\beta_2}}$

$\Rightarrow$ $\frac{pb}{qa}=\sqrt{\frac{pc}{ra}}\Rightarrow\frac{b^2}{q^2}=\frac{ac}{pr}$

4. Since $\begin{vmatrix}3-x & -6 & 3\\ -6 & 3-x & 3\\ 3 & 3 & -6-x\end{vmatrix}=0$,

applying $C_1 \to C_1 + C_2 + C_3$, we obtain

$\Rightarrow$ $-x\begin{vmatrix}1 & -6 & 3\\ 1 & 3-x & 3\\ 1 & 3 & -6-x\end{vmatrix}=0$

Applying $R_2 \to R_2 - R_1; R_3 \to R_3 - R_1$, we get

$$-x\begin{vmatrix}1 & -6 & 3\\ 0 & 9-x & 0\\ 0 & 9 & -9-x\end{vmatrix}=0$$

$\Rightarrow$ $-x(9-x)(-9-x)=0$

$\therefore$ $x = 0, 9, -9$

5. We have $\Delta^2=\Delta\cdot\Delta=\begin{vmatrix}l_1 & m_1 & n_1\\ l_2 & m_2 & n_2\\ l_3 & m_3 & n_3\end{vmatrix}\cdot\begin{vmatrix}l_1 & m_1 & n_1\\ l_2 & m_2 & n_2\\ l_3 & m_3 & n_3\end{vmatrix}$

$$=\begin{vmatrix}l_1^2+m_1^2+n_1^2 & l_1l_2+m_1m_2+n_1n_2 & l_1l_3+m_1m_3+n_1n_3\\ l_1l_2+m_1m_2+n_1n_2 & l_2^2+m_2^2+n_2^2 & l_2l_3+m_2m_3+n_2n_3\\ l_1l_3+m_1m_3+n_1n_3 & l_2l_3+m_2m_3+n_2n_3 & l_3^2+m_3^2+n_3^2\end{vmatrix}$$

$$=\begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix}=1 \quad\Rightarrow\quad \Delta=\pm 1$$

$\Rightarrow$ $|\Delta| = 1$

6. Operating $C_1 \to C_1 + C_2 + C_3$, the given determinant

$$=\begin{vmatrix}1+\omega+\omega^2 & \omega & \omega^2\\ \omega+\omega^2+1 & \omega^2 & 1\\ \omega^2+1+\omega & 1 & \omega\end{vmatrix}=\begin{vmatrix}0 & \omega & \omega^2\\ 0 & \omega^2 & 1\\ 0 & 1 & \omega\end{vmatrix}=0$$

7. Let $\Delta=\begin{vmatrix}1 & \sin 3\theta & \sin^3\theta\\ 2\cos\theta & \sin 6\theta & \sin^3 2\theta\\ 4\cos^2\theta-1 & \sin 9\theta & \sin^3 3\theta\end{vmatrix}$

Multiplying C_1 by $\sin\theta$, the determinant can be written as

$$\Delta=\frac{1}{\sin\theta}\begin{vmatrix}\sin\theta & \sin 3\theta & \sin^3\theta\\ 2\cos\theta\sin\theta & \sin 6\theta & \sin^3 2\theta\\ \sin\theta(4\cos^2\theta-1) & \sin 9\theta & \sin^3 3\theta\end{vmatrix}$$

$$=\frac{1}{\sin\theta}\begin{vmatrix}\sin\theta & \sin 3\theta & \sin^3\theta\\ \sin 2\theta & \sin 6\theta & \sin^3 2\theta\\ \sin 3\theta & \sin 9\theta & \sin^3 3\theta\end{vmatrix}$$

Now applying $C_1 \to 3C_1 - 4C_3$, we get

$$\Delta=\frac{1}{\sin\theta}\begin{vmatrix}3\sin\theta-\sin^3\theta & \sin 3\theta & \sin^3\theta\\ 3\sin 2\theta-4\sin^3 2\theta & \sin 6\theta & \sin^3 2\theta\\ 3\sin 3\theta-4\sin^3 3\theta & \sin 9\theta & \sin^3 3\theta\end{vmatrix}$$

$$\Delta=\frac{1}{\sin\theta}\begin{vmatrix}\sin 3\theta & \sin 3\theta & \sin^3\theta\\ \sin 6\theta & \sin 6\theta & \sin^3 2\theta\\ \sin 9\theta & \sin 9\theta & \sin^3 3\theta\end{vmatrix}=0$$

8. Let $\Delta=\begin{vmatrix}b(b-a) & b-c & c(b-a)\\ a(b-a) & a-b & b(b-a)\\ c(b-a) & c-a & a(b-a)\end{vmatrix}$

$\Rightarrow$ $\Delta=(b-a)^2\begin{vmatrix}b & b-c & c\\ a & a-b & b\\ c & c-a & a\end{vmatrix}$

Applying $C_1 \to C_1 - C_2 + C_3$, we get

$$\therefore \quad \Delta = (b-a)^2 \begin{vmatrix} 0 & b-c & c \\ 0 & a-b & b \\ 0 & c-a & a \end{vmatrix} = 0$$

9. Here $-(x-a)[-(x+b)(x-c)]$

$+(x-b)[(x+a)(x-c)] = 0$

$\Rightarrow \quad (x-a)(x+b)(x-c) + (x+a)(x-b)(x+c) = 0$

$\Rightarrow \quad x = 0$ [$\because$ there will be no constant term]

10. $\begin{vmatrix} bc & ca & ab \\ ca & ab & bc \\ ab & bc & ca \end{vmatrix} = 0$

$\Rightarrow \quad 3a^2b^2c^2 - [(ab)^3 + (bc)^3 + (ca)^3] = 0$

$\Rightarrow \quad (ab)^3 + (bc)^3 + (ca)^3 - 3a^2b^2c^2 = 0$

We know that

if $x^3 + y^3 + z^3 = 3xyz$, then $x + y + z = 0$

$\therefore \quad ab + bc + ca = 0$

$\Rightarrow \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$

11. Here the given determinant is

$$= \begin{vmatrix} 0 & \sin B & \cos C \\ -\sin B & 0 & \tan A \\ \cos(\pi - C) & -\tan A & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \sin B & \cos C \\ -\sin B & 0 & \tan A \\ -\cos C & -\tan A & 0 \end{vmatrix} = 0.$$

[$\because$ Determinant is skew-symmetric of odd order]

12. Let $\Delta = \begin{vmatrix} x & p & q \\ p & x & q \\ p & q & x \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we obtain

$$\Rightarrow \quad \Delta = \begin{vmatrix} x+p+q & p & q \\ x+p+q & x & q \\ x+p+q & q & x \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (x+p+q)\begin{vmatrix} 1 & p & q \\ 1 & x & q \\ 1 & q & x \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$, we get

$$\Rightarrow \quad \Delta = (x+p+q)\begin{vmatrix} 1 & p & q \\ 0 & x-p & 0 \\ 0 & q-p & x-q \end{vmatrix}$$

Expanding along C_1, we have

$$\Delta = (x+p+q)(x-p)(x-q)$$

13. Operating $R_3 \to R_3 - \alpha R_1 - R_2$,

$$\begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ 0 & 0 & -(a\alpha^2 + 2b\alpha + c) \end{vmatrix} = 0$$

$\Rightarrow \quad -(a\alpha^2 + 2b\alpha + c)(ac - b^2) = 0$

$\Rightarrow \quad b^2 = ac$ or $a\alpha^2 + 2b\alpha + c = 0$

$\Rightarrow \quad a, b, c$ are in G.P. or α is a root of $ax^2 + 2bx + c = 0$.

14. $f\left(\frac{\pi}{3}\right) = \begin{vmatrix} 2\cos\frac{\pi}{3} & 1 & 0 \\ 1 & 2\cos\frac{\pi}{3} & 1 \\ 0 & 1 & 2\cos\frac{\pi}{3} \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}, \text{ Applying } R_2 \to R_2 - R_1$$

$$f\left(\frac{\pi}{3}\right) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -1$$

15. $\Delta = \begin{vmatrix} 1 & \frac{\log y}{\log x} & \frac{\log z}{\log x} \\ \frac{\log x}{\log y} & 1 & \frac{\log z}{\log y} \\ \frac{\log x}{\log z} & \frac{\log y}{\log z} & 1 \end{vmatrix}$

$$= \frac{1}{\log x \log y \log z}\begin{vmatrix} \log x & \log y & \log z \\ \log x & \log y & \log z \\ \log x & \log y & \log z \end{vmatrix} = 0$$

16. Let $\Delta = \begin{vmatrix} ka & k^2+a^2 & 1 \\ kb & k^2+b^2 & 1 \\ kc & k^2+c^2 & 1 \end{vmatrix}$

$$\Rightarrow \quad \Delta = \begin{vmatrix} ka & k^2 & 1 \\ kb & k^2 & 1 \\ kc & k^2 & 1 \end{vmatrix} + \begin{vmatrix} ka & a^2 & 1 \\ kb & b^2 & 1 \\ kc & c^2 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = 0 + k\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$\therefore \quad \Delta = k(a-b)(b-c)(c-a)$

17. $$\sum_{k=1}^{n} D_k = \begin{vmatrix} \sum_{k=1}^{n} 1 & n & n \\ 2\sum_{k=1}^{n} k & n^2+n+1 & n^2+n \\ 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 & n^2 & n^2+n+1 \end{vmatrix}$$

$$= \begin{vmatrix} n & n & n \\ n^2+n & n^2+n+1 & n^2+n \\ n^2 & n^2 & n^2+n+1 \end{vmatrix}$$

Operating $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_2$

$$= \begin{vmatrix} n & 0 & 0 \\ n^2+n & 1 & -1 \\ n^2 & 0 & n+1 \end{vmatrix}$$

$= n(n+1)$.

By the equation, $n(n+1) = 56 = 7 \cdot 8 \Rightarrow n = 7$.

18. $$\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$$

$$= \begin{vmatrix} b^2+c^2 & ab & ca \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$$

19. Since $\Delta = \begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$

applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\Rightarrow \Delta = (a+b+c-x)\begin{vmatrix} 1 & 1 & 1 \\ c & b-x & a \\ b & a & c-x \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$; $C_3 \to C_3 - C_1$

$$\Rightarrow \Delta = (-x)\begin{vmatrix} 1 & 0 & 0 \\ c & b-x-c & a-c \\ b & a-b & c-x-b \end{vmatrix} \quad [\because (a+b+c=0)]$$

$\Rightarrow \Delta = -x\{(b-x-c)(c-x-b)-(a-b)(a-c)\}$

$\Rightarrow \Delta = x(a^2+b^2+c^2-ab-bc-ca-x^2)$

If $\Delta = 0 \Rightarrow x = 0$

or $x^2 = a^2+b^2+c^2-ab-bc-ca$

$\Rightarrow x^2 = a^2+b^2+c^2 - \frac{1}{2}\{(a+b+c)^2 - a^2 - b^2 - c^2\}$

$\Rightarrow x^2 = \frac{3}{2}(a^2+b^2+c^2) \quad [\because a+b+c=0]$

$\therefore x = \pm\sqrt{\frac{3}{2}(a^2+b^2+c^2)}$

20. $$\Delta = (2\cos x + \sin x)\begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix}$$

$(C_1 \to C_1 + C_2 + C_3)$

$$= (2\cos x + \sin x)\begin{vmatrix} 0 & \cos x - \sin x & 0 \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix}$$

$(R_1 \to R_1 - R_2)$

$= (2\cos x + \sin x)(\cos x - \sin x)^2 = 0$

$\Rightarrow \tan x = -2$ or 1, hence only one solution in $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$

21. Since $\begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = 0$,

applying $C_1 \to C_1 + (C_2 + C_3)$, we get

$$\Rightarrow \begin{vmatrix} x+9 & 3 & 5 \\ x+9 & x+2 & 5 \\ x+9 & 3 & x+4 \end{vmatrix} = 0$$

$$\Rightarrow (x+9)\begin{vmatrix} 1 & 3 & 5 \\ 1 & x+2 & 5 \\ 1 & 3 & x+4 \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$, we get

$$\Rightarrow (x+9)\begin{vmatrix} 1 & 3 & 5 \\ 0 & x-1 & 0 \\ 0 & 0 & x-1 \end{vmatrix} = 0$$

$\Rightarrow (x+9)(x-1)(x-1) = 0$

$\therefore x = -9, 1, 1$

22. $$\begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ 0 & xp+y & yp+z \end{vmatrix} = 0$$

$\Rightarrow y^2 = xz \Rightarrow x, y, z$ are in G.P.

23. $\Delta = \begin{vmatrix} {}^{10}C_4 & {}^{10}C_5 & {}^{11}C_m \\ {}^{11}C_6 & {}^{11}C_7 & {}^{12}C_{m+2} \\ {}^{12}C_8 & {}^{12}C_9 & {}^{13}C_{m+4} \end{vmatrix} = 0$, Applying $C_2 \to C_1 + C_2$

$$\Delta = \begin{vmatrix} {}^{10}C_4 & {}^{10}C_4 + {}^{10}C_5 & {}^{11}C_m \\ {}^{11}C_6 & {}^{11}C_6 + {}^{11}C_7 & {}^{12}C_{m+2} \\ {}^{12}C_8 & {}^{12}C_8 + {}^{12}C_9 & {}^{13}C_{m+4} \end{vmatrix} = 0$$

$$= \begin{vmatrix} {}^{10}C_4 & {}^{11}C_5 & {}^{11}C_m \\ {}^{11}C_6 & {}^{12}C_7 & {}^{12}C_{m+2} \\ {}^{12}C_8 & {}^{13}C_9 & {}^{13}C_{m+4} \end{vmatrix} = 0$$

Clearly, $m = 5$ satisfies the above result.

($\because$ C_2, C_3 will be identical)

24. Applying $C_1 \to C_1 + C_2$

$$f(x) = \begin{vmatrix} 2 & \cos^2 x & 4\sin 2x \\ 2 & 1+\cos^2 x & 4\sin 2x \\ 1 & \cos^2 x & 1+4\sin 2x \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, we have

$$f(x) = \begin{vmatrix} 2 & \cos^2 x & 4\sin 2x \\ 0 & 1 & 0 \\ 1 & \cos^2 x & 1+4\sin 2x \end{vmatrix}$$

$\Rightarrow$ $f(x) = 2 + 4 \sin 2x$

The value of $f(x)$ is maximum when $\sin 2x = 1$.

$\therefore$ Maximum value of $f(x) = 6$

25. Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$ in succession, we get

$$\Delta = \begin{vmatrix} \alpha-\beta & \beta-\gamma & \gamma \\ 0 & 0 & 1 \\ \beta-\gamma & \gamma-\alpha & \alpha \end{vmatrix}$$

$\therefore$ $\Delta = \alpha^2 + \beta^2 + \gamma^2 - (\alpha\beta + \beta\gamma + \gamma\alpha)$

$= (\alpha+\beta+\gamma)^2 - 3(\alpha\beta + \beta\gamma + \gamma\alpha) = p^2 - 3q$

26. Let $\Delta = \begin{vmatrix} {}^5C_0 & {}^5C_3 & 1 \\ {}^5C_1 & {}^5C_4 & 1 \\ {}^5C_2 & {}^5C_5 & 1 \end{vmatrix}$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 10 & 1 \\ 5 & 5 & 1 \\ 10 & 1 & 1 \end{vmatrix}$$

Applying $R_3 \to R_3 - R_1$ and $R_2 \to R_2 - R_1$, we get

$$\Delta = \begin{vmatrix} 1 & 10 & 1 \\ 4 & -5 & 0 \\ 9 & -9 & 0 \end{vmatrix}$$

$\therefore$ $\Delta = -36 + 45 = 9$

27. $\begin{vmatrix} a+pd & a+qd & a+rd \\ p & q & r \\ d & d & d \end{vmatrix}$

$$= \begin{vmatrix} a & a & a \\ p & q & r \\ d & d & d \end{vmatrix} + d\begin{vmatrix} p & q & r \\ p & q & r \\ d & d & d \end{vmatrix} = 0 + d(0) = 0$$

28. $\Delta = \begin{vmatrix} {}^5C_0 & {}^5C_3 & 14 \\ {}^5C_1 & {}^5C_4 & 1 \\ {}^5C_2 & {}^5C_5 & 1 \end{vmatrix}$ (Operating $R_1 \to R_1 + R_2 + R_3$)

$$= \begin{vmatrix} {}^5C_0 + {}^5C_1 + {}^5C_2 & {}^5C_3 + {}^5C_4 + {}^5C_5 & 14+1+1 \\ 5 & 5 & 1 \\ 10 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 16 & 16 & 16 \\ 5 & 5 & 1 \\ 10 & 1 & 1 \end{vmatrix} = 16\begin{vmatrix} 1 & 1 & 1 \\ 5 & 5 & 1 \\ 10 & 1 & 1 \end{vmatrix}$$

$[\because {}^5C_0 + {}^5C_1 + {}^5C_2 = {}^5C_3 + {}^5C_4 + {}^5C_5 = 16]$

$$= 16\begin{vmatrix} 1 & 1 & 1 \\ 5 & 5 & 1 \\ 9 & 0 & 0 \end{vmatrix} = 16 \times 9(-4) = -576$$

29. $\Delta'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$

$$+\begin{vmatrix} f(x) & g(x) & h(x) \\ f''(x) & g''(x) & h''(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ 0 & 0 & 0 \end{vmatrix}$$

$\Rightarrow$ $\Delta'(x) = 0$

$\Rightarrow$ $\Delta(x)$ is constant, i.e. polynomial of degree 0.

30. $C_1 \to C_1 + C_2 + C_3$

$$\Delta = \begin{vmatrix} 1+(1+\omega+\omega^2) & \omega & \omega^2 \\ 1+(1+\omega+\omega^2) & -\omega & \omega \\ 1+(1+\omega+\omega^2) & 1 & -\omega^2 \end{vmatrix}$$

$R_1 \to R_1 - R_2$; $R_2 \to R_2 - R_3$

$$\Delta = \begin{vmatrix} 0 & 2\omega & \omega^2-\omega \\ 0 & -\omega-1 & \omega+\omega^2 \\ 1 & 1 & -\omega^2 \end{vmatrix}$$

$= 2\omega(\omega + \omega^2) - (\omega^2 - \omega)(-\omega - 1)$

$= -2\omega - \omega + 1 = 1 - 3\omega$

31. Differentiating w.r.t. x, we get

$$-f'(x) = \begin{vmatrix} -\sin x & x & 1 \\ 2\cos x & x^2 & 2x \\ \sec^2 x & x & 1 \end{vmatrix} + \begin{vmatrix} \cos x & 1 & 1 \\ 2\sin x & 2x & 2x \\ \tan x & 1 & 1 \end{vmatrix}$$

$$+\begin{vmatrix} \cos x & x & 0 \\ 2\sin x & x^2 & 2 \\ \tan x & x & 0 \end{vmatrix}$$

$$\Rightarrow \quad -\frac{f'(x)}{x} = \begin{vmatrix} -\sin x & 1 & 1 \\ 2\cos x & x & 2x \\ \sec^2 x & 1 & 1 \end{vmatrix} + \begin{vmatrix} \cos x & 1 & 0 \\ 2\sin x & x & 2 \\ \tan x & 1 & 0 \end{vmatrix}$$

[$\because$ the second determinant is zero]

$$\Rightarrow \quad -\lim_{x\to 0}\frac{f'(x)}{x} = \begin{vmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{vmatrix} = -2$$

$$\Rightarrow \quad \lim_{x\to 0}\frac{f'(x)}{x} = 2$$

32. a, b, c are in A.P. $\Rightarrow \quad 2b = a + c$ (i)

The given determinant is

$$= \frac{1}{2}\begin{vmatrix} x+1 & x+2 & x+a \\ 2x+4 & 2x+6 & 2x+2b \\ x+3 & x+4 & x+c \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} x+1 & x+2 & x+a \\ 0 & 0 & 2b-(a+c) \\ x+3 & x+4 & x+6 \end{vmatrix}$$

[Applying $R_2 \to R_2 - (R_1 + R_3)$]

$$= \frac{1}{2}\begin{vmatrix} x+1 & x+2 & x+a \\ 0 & 0 & 0 \\ x+3 & x+4 & x+6 \end{vmatrix} = 0 \quad \text{[Using (i)]}$$

33. $$\Delta(x) = \begin{vmatrix} x & 1+x^2 & x^3 \\ \log(1+x^2) & e^x & \sin x \\ \cos x & \tan x & \sin^2 x \end{vmatrix}$$

Let $\Delta(x) = A + Bx + Cx^2 + Dx^3 + \cdots$

$$\Delta(0) = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0 \quad \Rightarrow \quad A = 0$$

$$\therefore \quad \Delta'(x) = \begin{vmatrix} 1 & 1+x^2 & x^3 \\ \frac{2x}{1+x^2} & e^x & \sin x \\ -\sin x & \tan x & \sin^2 x \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & 2x & x^3 \\ \log(1+x^2) & e^x & \sin x \\ \cos x & \sec^2 x & \sin^2 x \end{vmatrix}$$

$$+ \begin{vmatrix} x & 1+x^2 & 3x^2 \\ \log(1+x^2) & e^x & \cos x \\ \cos x & \tan x & 2\sin x\cos x \end{vmatrix}$$

$$\therefore \quad \Delta'(0) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

$= 0 + 0 + 1 = 1$

Also, $\Delta'(x) = B + 2xC + 3x^2 D \quad \therefore \quad \Delta'(0) = B$

$\therefore \quad B = 1 \quad \therefore \quad \Delta(x) = x + Cx^2 + Dx^3 + \cdots$

$\therefore \quad \Delta(x)$ is divisible by x.

34. $$\frac{d^n}{dx^n}\Delta(x) = \begin{vmatrix} \frac{d^n}{dx^n}x^n & \frac{d^n}{dx^n}(\sin x) & \frac{d^n}{dx^n}(\cos x) \\ n! & \sin\frac{n\pi}{2} & \cos\frac{n\pi}{2} \\ \alpha & \alpha^2 & \alpha^3 \end{vmatrix}$$

$$\frac{d^n}{dx^n}\Delta(x) = \begin{vmatrix} n! & \sin\left(x+\frac{n\pi}{2}\right) & \cos\left(x+\frac{n\pi}{2}\right) \\ n! & \sin\frac{n\pi}{2} & \cos\frac{n\pi}{2} \\ \alpha & \alpha^2 & \alpha^3 \end{vmatrix}$$

$$\therefore \quad \left(\frac{d^n}{dx^n}\Delta(x)\right)_{x=0} = \begin{vmatrix} n! & \sin\frac{n\pi}{2} & \cos\frac{n\pi}{2} \\ n! & \sin\frac{n\pi}{2} & \cos\frac{n\pi}{2} \\ \alpha & \alpha^2 & \alpha^3 \end{vmatrix} = 0$$

35. Putting $x = 0$, we have

$$f = \begin{vmatrix} 3 & -5 & -2 \\ 0 & 2 & 1 \\ 0 & 7 & 1 \end{vmatrix} = 3(2-7) = -15$$

36. Since $\begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix} = 0,$

Expanding the given determinant, we have

$$2x^3 + 2x(ac - ab - bc) = 0$$

$\therefore \quad x = 0$

37. Let $s - a = A, s - b = B, s - c = C.$

$\therefore \quad A + B + C = 3s - (a+b+c) = 3s - 2s = s$

$B + C = 2s - (b+c) = a + b + c - (b+c) = a.$

Similarly, $C + A = b, A + B = c$

$$\therefore \quad \Delta = \begin{vmatrix} a^2 & (s-a)^2 & (s-a)^2 \\ (s-b)^2 & b^2 & (s-b)^2 \\ (s-c)^2 & (s-c)^2 & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} (B+C)^2 & A^2 & A^2 \\ B^2 & (C+A)^2 & B^2 \\ C^2 & C^2 & (A+B)^2 \end{vmatrix}$$

Operating $C_1 \to C_1 - C_3, C_2 \to C_2 - C_3$

$$\Delta = \begin{vmatrix} (B+C)^2 - A^2 & 0 & A^2 \\ 0 & (C+A)^2 - B^2 & B^2 \\ C^2 - (A+B)^2 & C^2 - (A+B)^2 & (A+B)^2 \end{vmatrix}$$

$$= (A+B+C)^2 \begin{vmatrix} B+C-A & 0 & A^2 \\ 0 & C+A-B & B^2 \\ C-A-B & C-A-B & (A+B)^2 \end{vmatrix}$$

Operating $R_3 \to R_3 - (R_1 + R_2)$,

$$\Delta = (A+B+C)^2 \begin{vmatrix} B+C-A & 0 & A^2 \\ 0 & C+A-B & B^2 \\ -2B & -2A & 2AB \end{vmatrix}$$

$$= 2(A+B+C)^2 \begin{vmatrix} B+C-A & 0 & A^2 \\ 0 & C+A-B & B^2 \\ -B & -A & AB \end{vmatrix}$$

$$= 2(A+B+C)^2\{[(B+C-A)]$$
$$[(AB)(C+A-B)+AB^2]+A^2[0+B(C+A-B)]\}$$
$$= 2AB(A+B+C)^2[(B+C-A)(C+A-B+B)$$
$$+A(C+A-B)]$$
$$= 2AB(A+B+C)^2[BC+C^2-CA+AB$$
$$+AC-A^2+AC+A^2-AB]$$
$$= 2AB(A+B+C)^2[BC+C^2+AC]$$
$$= 2ABC(A+B+C)^3$$
$$= 2s^3(s-a)(s-b)(s-c)$$

38. Applying $C_1 \to C_1 + C_2 + C_3$, we have

$$\Delta = (\alpha\beta + \beta\gamma + \gamma\alpha) \begin{vmatrix} 1 & \beta\gamma & \gamma\alpha \\ 1 & \gamma\alpha & \alpha\beta \\ 1 & \alpha\beta & \beta\gamma \end{vmatrix}$$

Also, since $\alpha, \beta, \gamma \in px^3 + qx^2 + r = 0$,

$$\therefore \quad S_2 = \alpha\beta + \beta\gamma + \gamma\alpha = (-1)^2 \frac{0}{p} = 0$$

Hence $\Delta = 0$

39. $\int f(x)dx = \begin{vmatrix} x & x^2 & x^3 \\ 3 & a & 27 \\ 1 & 3 & 9 \end{vmatrix}$

$$\therefore \int_0^3 f(x)dx = \begin{vmatrix} 3 & 9 & 27 \\ 3 & a & 27 \\ 1 & 3 & 9 \end{vmatrix} - \begin{vmatrix} 0 & 0 & 0 \\ 3 & a & 27 \\ 1 & 3 & 9 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 9 & 27 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Hence, a can be any real number.

40. Differentiating both sides w.r.t. x, we have

$$\begin{vmatrix} 1 & 1 & 2x \\ x & 1+x & x^2 \\ x^2 & x & 1+x \end{vmatrix} + \begin{vmatrix} 1+x & x & x^2 \\ 1 & 1 & 2x \\ x^2 & x & 1+x \end{vmatrix} + \begin{vmatrix} 1+x & x & x^2 \\ x & 1+x & x^2 \\ 2x & 1 & 1 \end{vmatrix}$$

$$= 5ax^4 + 4bx^3 + 3cx^2 + 2dx + \lambda$$

Putting $x = 0$, we have

$$\lambda = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

$\Rightarrow \quad \lambda = 1 + 1 + 1 = 3$

41. Taking a common from R_1 and C_1, b from R_2 and C_2, c from R_3 and C_3, we have

$$\text{L.H.S.} = a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

Applying $R_2 \to R_2 + R_1, R_3 \to R_3 + R_1$,

$$= a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix} = 4a^2b^2c^2$$

$\therefore \quad \lambda = 4$

42. Applying $R_1 \to R_1 + R_2 + R_3$, we get the given determinant

$$\begin{vmatrix} 3+a & 3+a & 3+a \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$$

$$= (3+a) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$$

Applying $C_3 \to C_3 - C_1, C_2 \to C_2 - C_1$

$$= (3+a)\begin{vmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix} = (3+a)a^2$$

$$= a^3\left(1+\frac{3}{a}\right)$$

43. Since $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = 64,$

applying $R_2 \to R_2 - 2R_1; R_3 \to R_3 - 3R_1$, we get

$$\begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 3a & 7a+3b \end{vmatrix} = 64$$

Expanding along C_1, we have

$$a(7a^2 + 3ab - 6a^2 - 3ab) = 64$$

$\Rightarrow \quad a(a^2) = 64$

$\therefore \quad a = 4$

44. $\Delta = \begin{vmatrix} 1 & \frac{\log y}{\log x} & \frac{\log z}{\log x} \\ \frac{\log x}{\log y} & 2 & \frac{\log z}{\log y} \\ \frac{\log x}{\log z} & \frac{\log y}{\log z} & 4 \end{vmatrix}$

$$\Rightarrow \quad \Delta = \frac{1}{\log x \log y \log z}\begin{vmatrix} \log x & \log y & \log z \\ \log x & 2\log y & \log z \\ \log x & \log y & 4\log z \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \frac{1}{\log x \log y \log z}\begin{vmatrix} \log x & \log y & \log z \\ 0 & \log y & 0 \\ 0 & 0 & 3\log z \end{vmatrix}$$

$$= \frac{3\log x \log y \log z}{\log x \log y \log z} = 3 \text{ (diagonal property)}$$

45. $\Delta = \begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix}$

$$= \begin{vmatrix} a^2 & b^2 & c^2 \\ 4a & 4b & 4c \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix}$$

[Applying $R_2 \to R_2 - R_3$]

$$= 4\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix}$$

$$= 4\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1-2a & 1-2b & 1-2c \end{vmatrix} \quad \text{[Applying } R_3 \to R_3 - R_1\text{]}$$

Applying $R_3 \to R_3 + 2R_2$

$$= 4\begin{vmatrix} a^2 & b^2-a^2 & c^2-a^2 \\ a & b-a & c-a \\ 1 & 0 & 0 \end{vmatrix}$$

$$= 4\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

$= 4(b-a)(b-c)(c-a) = 0$

Either $b - a = 0$ or $b - c = 0$ or $c - a = 0$

Either $a = b$ or $b = c$ or $c = a$

i.e. ΔABC is isosceles.

46. $\sum 2^{r-1} = 1 + 2 + 2^2 + \cdots + 2^{n-1} = 1 \cdot \frac{2^{n-1}}{2-1} = 2^{n-1}$

$\sum 2 \cdot 3^{r-1} = 2(1 + 3 + 3^2 + \cdots + 3^{n-1}) = \frac{2(3^{n-1})}{3-1} = 3^{n-1}$

$\sum 4 \cdot 5^{r-1} = 4(1 + 5 + 5^2 + \cdots + 5^{n-1}) = \frac{4(5^{n-1})}{5-1} = 5^{n-1}$

$$\therefore \quad \sum D_r = \begin{vmatrix} 2^{n-1} & 3^{n-1} & 5^{n-1} \\ \alpha & \beta & \gamma \\ 2^{n-1} & 3^{n-1} & 5^{n-1} \end{vmatrix} = 0$$

[$\because R_1$ and R_3 are identical]

47. $\Delta = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = 0$

$$\Rightarrow \quad (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & a & b \\ 1 & c & a \end{vmatrix} = 0 \; (C_1 \to C_1 + C_2 + C_3)$$

48. We have

$$f(-x) = \begin{vmatrix} -x^3 & \cos^2 x & 2^{x^4} \\ -\tan^5 x & 1 & \sec 2x \\ -\sin^3 x & x^4 & 5 \end{vmatrix} = -f(x)$$

$$\therefore \quad \int_{\pi/2}^{\pi/2} f(x)dx = 0$$

49. Differentiating w.r.t. x, we get

$$f'(x) = (1+2x)\begin{vmatrix} -\sin(x+x^2) & \cos(x+x^2) & \sin(x+x^2) \\ \sin(x-x^2) & \cos(x-x^2) & \sin(x-x^2) \\ \sin 2x & 0 & \sin(2x^2) \end{vmatrix}$$

$$+(1-2x)\begin{vmatrix} \cos(x+x^2) & \sin(x+x^2) & -\cos(x+x^2) \\ \cos(x-x^2) & -\sin(x-x^2) & \cos(x-x^2) \\ \sin 2x & 0 & \sin(2x^2) \end{vmatrix}$$

$$+2\begin{vmatrix} \cos(x+x^2) & \sin(x+x^2) & -\cos(x+x^2) \\ \sin(x-x^2) & \cos(x-x^2) & \sin(x-x^2) \\ \cos(2x) & 0 & 2x\cos(2x^2) \end{vmatrix}$$

$$\Rightarrow \quad f'(0) = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix} + 2\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= 0 + 0 + 2(1) = 2$$

50. $\begin{vmatrix} a & 1 & 4 \\ 4 & a & 8 \\ 2 & 1 & 2a \end{vmatrix} = 0$

$\Rightarrow \quad 2a^3 - 24a + 32 = 0$

$\Rightarrow \quad 2(a-2)(a^2+2a-8) = 0$

$\Rightarrow \quad a = 2, -4$

51. Since $f(x) = \begin{vmatrix} \sec^2 x & 1 & 1 \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cot^2 x \end{vmatrix}$,

applying $C_2 \to C_2 - \cos^2 x C_1$, we get

$$\Rightarrow \quad f(x) = \begin{vmatrix} \sec^2 x & 0 & 1 \\ \cos^2 x & \cos^2 x - \cos^4 x & \operatorname{cosec}^2 x \\ 1 & 0 & \cot^2 x \end{vmatrix}$$

Expanding along C_2, we have

$$f(x) = (\cos^2 x - \cos^4 x)(\sec^2 x \cot^2 x - 1)$$

$\Rightarrow \quad f(x) = \cos^2 x \sin^2 x(\operatorname{cosec}^2 x - 1)$

$\therefore \quad f(x) = \cos^2 x \sin^2 x \cot^2 x = \cos^4 x$

52. $$\sum_{r=1}^{n} S_r = \begin{vmatrix} \sum_{r=1}^{n} 2r & x & n(n+1) \\ \sum_{r=1}^{n} (6r^2-1) & y & n^2(2n+3) \\ \sum_{r=1}^{n} (4r^3-2nr) & z & n^3(n+1) \end{vmatrix}$$

$$= \begin{vmatrix} n(n+1) & x & n(n+1) \\ n(n+1)(2n+1)-n & y & n^2(2n+3) \\ n^2(n+1)^2 - n^2(n+1) & z & n^3(n+1) \end{vmatrix}$$

$$= \begin{vmatrix} n(n+1) & x & n(n+1) \\ n^2(2n+3) & y & n^2(2n+3) \\ n^3(n+1) & z & n^3(n+1) \end{vmatrix} = 0$$

[$\because$ C_1 and C_3 are identical]

which is independent of x, y, z, n.

53. Since the the given system of equations has a non-trivial solution

$$\therefore \quad \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\begin{vmatrix} a & 1 & 1 \\ 1-a & b-1 & 0 \\ 1-a & 0 & c-1 \end{vmatrix} = 0$$

$\Rightarrow \quad a(b-1)(c-1) - (1-a)(c-1) - (1-a)(b-1) = 0$

Dividing by $(1-a)(1-b)(1-c)$, we get

$$\frac{a}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 0$$

$$\Rightarrow \quad \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$$

54. $\begin{vmatrix} 3 & k \\ k & k \end{vmatrix} = 0 \Rightarrow k = 0, 3$ (k = 0 is not possible)

$$\Delta_1 = \begin{vmatrix} k & 2 \\ k & 4 \end{vmatrix} \neq 0 \text{ if } k = 3$$

i.e. at k = 3, the system is inconsistent.

55. Since x, y, z are in A.P.,

$\Rightarrow \quad 2y = x + z$ (i)

Now, $$|A| = \begin{bmatrix} 4 & 5 & 6 & x \\ 5 & 6 & 7 & y \\ 6 & 7 & 8 & z \\ x & y & z & 0 \end{bmatrix}$$

Applying $R_2 \to 2R_2 - (R_1 + R_3)$, we get

$$\Rightarrow \quad |A| = \begin{bmatrix} 4 & 5 & 6 & x \\ 0 & 0 & 0 & 0 \\ 6 & 7 & 8 & 0 \\ x & y & z & 0 \end{bmatrix} \quad \text{[using (i)]}$$

$\therefore \quad |A| = 0$

56. $\sin 2x + 1 = 0 \quad \Rightarrow \quad 2x = \dfrac{\pi}{2} \Rightarrow x = \dfrac{\pi}{4}$

$$\therefore \begin{vmatrix} 0 & \cos x & -\sin x \\ \sin x & 0 & \cos x \\ \cos x & \sin x & 0 \end{vmatrix}^2 = \begin{vmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{vmatrix}^2$$

$$= \left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\right)^2 \begin{vmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}^2$$

$$= \frac{1}{8}[1(0-1) - 1(1-0)]^2 = \frac{1}{8}(-2)^2 = \frac{1}{2}$$

57. The given system of linear equations will have a non-trivial solution if

$$\begin{vmatrix} \lambda & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

Expanding the determinant along C_1, we get

$$\lambda(-\cos^2\alpha - \sin^2\alpha) - (-\sin\alpha\cos\alpha - \sin\alpha\cos\alpha)$$
$$-(\sin^2\alpha - \cos^2\alpha) = 0$$

$\Rightarrow \quad -\lambda + \sin 2\alpha + \cos 2\alpha = 0$

$\Rightarrow \quad \lambda = \sin 2\alpha + \cos 2\alpha = \sqrt{2}\sin(\pi/4 + 2\alpha)$

$\Rightarrow \quad -\sqrt{2} \le \lambda \le \sqrt{2}$

58. For a unique solution, $\Delta \ne 0$

$$\Rightarrow \begin{vmatrix} 2 & a & 6 \\ 1 & 2 & b \\ 1 & 1 & 3 \end{vmatrix} \ne 0 \Rightarrow \begin{vmatrix} 0 & a-2 & 0 \\ 0 & 1 & b-3 \\ 1 & 1 & 3 \end{vmatrix} \ne 0$$

$(R_1 \to R_1 - 2R_3, R_2 \to R_2 - R_3)$

$\Rightarrow \quad (a-2)(b-3) \ne 0 \quad \Rightarrow a \ne 2, b \ne 3$

59. Let $\Delta = \begin{bmatrix} x & x^2 - yz & 1 \\ y & y^2 - zx & 1 \\ z & z^2 - xy & 1 \end{bmatrix}$

$$\Rightarrow \quad \Delta = \begin{bmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{bmatrix} + \begin{bmatrix} x & -yz & 1 \\ y & -zx & 1 \\ z & -xy & 1 \end{bmatrix}$$

$$\Rightarrow \quad \Delta = \begin{bmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{bmatrix} - \frac{1}{xyz}\begin{bmatrix} x^2 & xyz & x \\ y^2 & zxy & y \\ z^2 & xyz & z \end{bmatrix}$$

$$\Rightarrow \quad \Delta = \begin{bmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{bmatrix} - \begin{bmatrix} x^2 & 1 & x \\ y^2 & 1 & y \\ z^2 & 1 & z \end{bmatrix}$$

In determinant II, interchanging the first $C_3 \leftrightarrow C_2$ and then $C_2 \leftrightarrow C_1$, we get

$$\Delta = \begin{bmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{bmatrix} - \begin{bmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{bmatrix} = 0$$

60. Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\begin{vmatrix} 2(x+y+z) & x+y+z & x+y+z \\ z+x & z & x \\ x+y & y & z \end{vmatrix}$$

$$= (x+y+z)\begin{vmatrix} 2 & 1 & 1 \\ z+x & z & x \\ x+y & y & z \end{vmatrix}$$

$$= (x+y+z)\begin{vmatrix} 0 & 1 & 1 \\ 0 & z & x \\ x-z & y & z \end{vmatrix}$$

[Operating $C_1 \to C_1 - C_2 - C_3$]

$$= (x+y+z)(x-z)^2$$

Hence, the repeated factor = $z - x$

61. A homogeneous system having infinite solutions means that it is non-trivial, i.e. $\Delta = 0$

$$\begin{vmatrix} \sin\theta & -\sqrt{3} \\ \cos\theta & 1 \end{vmatrix} = 0 \Rightarrow \tan\theta = -\sqrt{3} \Rightarrow \theta = \frac{4\pi}{3}$$

62. Since $p + q + r = 0 = a + b + c$ (given)

$\Rightarrow \quad p^3 + q^3 + r^3 = 3pqr$ or $a^3 + b^3 + c^3 = 3abc$

Let $\quad \Delta = \begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix}$

$\Rightarrow \quad \Delta = pqr(a^3 + b^3 + c^3) - abc(p^3 + q^3 + r^3)$

$\therefore \quad \Delta = pqr(3abc) - abc(3pqr) = 0$

63. $$\alpha = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix} = \frac{1}{xyz}\begin{vmatrix} x & x^2 & xyz \\ y & y^2 & xyz \\ z & z^2 & xyz \end{vmatrix}$$

$$= \frac{xyz}{xyz}\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = \beta$$

64. Let $\Delta = \begin{vmatrix} a^2+x & ab & ac \\ ab & b^2+x & bc \\ ac & bc & c^2+x \end{vmatrix}$

Applying $C_1 \to aC_1, C_2 \to bC_2, C_3 \to cC_3$, we get

$$\Delta = \frac{1}{abc}\begin{vmatrix} a^3+ax & a^2b & a^2c \\ ab^2 & b^3+bx & b^2c \\ ac^2 & bc^2 & c^3+cx \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} a^2+x & a^2 & a^2 \\ b^2 & b^2+x & b^2 \\ c^2 & c^2 & c^2+x \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\Rightarrow \quad \Delta = (a^2+b^2+c^2+x)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2+x & b^2 \\ c^2 & c^2 & c^2+x \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (a^2+b^2+c^2+x)\{(b^2x+c^2x+x^2) - (b^2x)+(-c^2x)\}$$

$$\Rightarrow \quad \Delta = x^2(a^2+b^2+c^2+x)$$

$\therefore$ x^2 is a factor of Δ as well as $a^2 + b^2 + c^2 + x$ is a factor of Δ.

HINTS AND SOLUTIONS—EXERCISE SET 2

1. Apply $C_3 \to C_3 + C_2$ and take $a + b + c$ common from C_3. $\therefore$ $\Delta = 0$ as two columns C_1 and C_3 are identical.

2. Let $\Delta = \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) & \cos 2\beta \\ \sin\alpha & \cos\alpha & \sin\beta \\ -\cos\alpha & \sin\alpha & \cos\beta \end{bmatrix}$

Applying $R_1 \to R_1 + R_2 \sin\beta + R_3 \cos\beta$, we get

$$\Delta = \begin{bmatrix} 0 & 0 & \cos 2\beta + 1 \\ \sin\alpha & \cos\alpha & \sin\beta \\ -\cos\alpha & \sin\alpha & \cos\beta \end{bmatrix}$$

$\Rightarrow \quad \Delta = (\cos 2\beta + 1)(\sin^2\alpha + \cos^2\alpha)$

$\therefore \quad \Delta = (\cos 2\beta + 1)$ which is independent of α

3. $|A|_{\max} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 0 - (-1) + 1(1) = 2$

$\therefore \quad |A|_{\min} = -2$

4. $f'(x) = \begin{vmatrix} -\sin x & \sin x & \cos x \\ -2\sin 2x & \sin 2x & 2\cos 2x \\ -3\sin 3x & \sin 3x & 3\cos 3x \end{vmatrix} + \begin{vmatrix} \cos x & \cos x & \cos x \\ \cos 2x & 2\cos 2x & 2\cos 2x \\ \cos 3x & 3\cos 3x & 3\cos 3x \end{vmatrix} + \begin{vmatrix} \cos x & \sin x & -\sin x \\ \cos 2x & \sin 2x & -4\sin 2x \\ \cos 3x & \sin 3x & -9\sin 3x \end{vmatrix}$

$$\therefore \quad f'\left(\frac{\pi}{2}\right) = \begin{vmatrix} -1 & 1 & 0 \\ 0 & 0 & -2 \\ 3 & -1 & 0 \end{vmatrix} + 0 + \begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 9 \end{vmatrix}$$

$= 2(1 - 3) + 0 + 1 \cdot (9 - 1)$

$= -4 + 8 = 4$

5. The determinant of coefficients of x, y, z is

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 5 & 5 & 9 \end{vmatrix}$$

$\Rightarrow \quad \Delta = 1(9-15) - 2(18-15) + 3(10-5)$

$\Rightarrow \quad \Delta = -6 - 6 + 15 = 3 \neq 0$

$\therefore$ The system of equations represents a unique solution.

6. $\log_x y = \dfrac{\log y}{\log x}$

Multiply R_1, R_2, R_3 by $\log x$, $\log y$ and $\log z$, respectively and divide Δ by $\log x \log y \log z$.
$\Delta = 0$ as all rows become identical.

7. $\begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix}$

$= pa(a^2qr - p^2bc) - qb(q^2ca - b^2pr) + rc(pqc^2 - r^2ab)$

$= pqr(a^3 + b^3 + c^3) - abc(p^3 + q^3 + r^3)$

$= pqr(a^3 + b^3 + c^3) - abc(3pqr)$

$[\because p + q + r = 0, p^3 + q^3 + r^3 = 3pqr]$

$= pqr(a^3 + b^3 + c^3 - 3abc)$

$$= pqr\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

8. Since the given system of equations has a non-trivial solution, we have

$$\Delta = \begin{vmatrix} a & 4 & 1 \\ b & 3 & 1 \\ c & 2 & 1 \end{vmatrix} = 0$$

[Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$]

$$\Rightarrow \quad \Delta = \begin{vmatrix} a-b & 1 & 0 \\ b-c & 1 & 0 \\ c & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad (a-b)-(b-c)=0$

$\Rightarrow \quad a-b=b-c$

or, $b = \dfrac{a+c}{2}$

$\therefore$ a, b, c are in A.P.

9. $\Delta = \Delta_1 - \Delta_2 = 0$ as $\Delta_2 = \Delta_1$

$$\Delta_2 = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$$

$$= \frac{abc}{abc}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = \Delta_1$$

10. $$\Delta = \begin{vmatrix} 1 & 2 & 6 \\ 2 & 6 & 24 \\ 6 & 24 & 120 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 6 \\ 0 & 2 & 12 \\ 0 & 12 & 84 \end{vmatrix}$$

[Operating $R_3 \to R_3 - 6R_1, R_2 \to R_2 - 2R_1$]

$= 168 - 144 = 24 = 4!$

11. The value of x, y, z are non-trivial if $\begin{vmatrix} 3 & k & -2 \\ 1 & k & 3 \\ 2 & 3 & -4 \end{vmatrix} = 0$

$\Rightarrow \quad 3(-4k-9) - 1(-4k+6) + 2(3k+2k) = 0$

$\Rightarrow \quad -12k - 27 + 4k - 6 + 10k = 0$

$\Rightarrow \quad 2k = 33 \quad \therefore \quad k = \dfrac{33}{2}$

12. Taking $(b-a)$ common from each of C_1 and C_3, we have

$$\Delta = (b-a)^2 \begin{vmatrix} b & b-c & c \\ a & a-b & b \\ c & c-a & a \end{vmatrix}$$

Now applying $C_1 \to C_1 - C_3$, the new C_1 and C_2 become identical

$\therefore \quad \Delta = 0$

13. $$\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$$

$$= \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} \qquad [\because a^2+b^2+c^2=0]$$

$$= abc\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix} = a^2b^2c^2\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= a^2b^2c^2\begin{vmatrix} 0 & 0 & 2 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 4a^2b^2c^2$$

Given: $4a^2b^2c^2 = ka^2b^2c^2 \quad \Rightarrow \quad k = 4$

14. For the system of equations to have non-trivial solutions,

$$\Delta = \begin{vmatrix} 2 & -1 & 1 \\ 1 & -2 & 1 \\ \lambda & -1 & 2 \end{vmatrix} = 0$$

$\Rightarrow \quad 2(-4+1) + 1(2-\lambda) + 1(-1+2\lambda) = 0$

$\Rightarrow \quad -6 + 2 - \lambda - 1 + 2\lambda = 0$

$\therefore \quad \lambda = 5$

15. $a = p$th term of H.P.

$\therefore \quad \dfrac{1}{a} = p$th term of A.P. $= A + (p-1)D$

$$\Delta = abc\begin{vmatrix} 1/a & 1/b & 1/c \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

[We have divided R_1 by abc and multiplied Δ by abc.]

$$= abc\begin{vmatrix} A+(p-1)D & A+(q-1)D & A+(r-1)D \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

$= \Delta_1 + \Delta_2$

Each of them is zero because of identical rows.

16. $$\Delta = \begin{vmatrix} a^3 & b^3 & c^3 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} + \begin{vmatrix} -1 & -1 & -1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$= abc\begin{vmatrix} a^2 & b^2 & c^2 \\ 1 & 1 & 1 \\ a & b & c \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$= (abc-1)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$= (abc-1)(a-b)(b-c)(c-a)$

$\therefore \quad \Delta = 0 \Rightarrow abc - 1 = 0 \qquad [\because a \neq b \neq c]$

$\Rightarrow \quad abc = 1$

17. The linear system of equations will possess a solution if

$$\begin{vmatrix} 2 & 3 & -8 \\ 7 & -5 & 3 \\ 4 & -6 & \lambda \end{vmatrix} = 0$$

$\Rightarrow \quad 2(-5\lambda+18) - 3(7\lambda-12) - 8(-42+20) = 0$

$\Rightarrow \quad -10\lambda + 36 - 21\lambda + 36 + 176 = 0$

$\Rightarrow \quad 31\lambda = 248$

$\therefore \quad \lambda = 8$

18. $\sum_{n=1}^{5} U_n = U_1 + U_2 + U_3 + U_4 + U_5$

Putting $n = 5$ in the formula for $\sum n, \sum n^2, \sum n^3$, we get

$$U_n = \begin{vmatrix} 15 & 15 & 8 \\ 55 & 35 & 9 \\ 225 & 25 & 10 \end{vmatrix} = \begin{vmatrix} 0 & 15 & 8 \\ 20 & 35 & 9 \\ 200 & 25 & 10 \end{vmatrix}$$

by applying $C_1 \to C_1 - C_2$

$$= 20 \times 5 \begin{vmatrix} 0 & 3 & 8 \\ 1 & 7 & 9 \\ 10 & 5 & 10 \end{vmatrix}$$

$= 100[-1(70-45) + 10(27-56)]$

$= 100(-25-290) = 100 \times -315$

$= -31500$

19. Differentiating both sides w.r.t. x, we get

$$\begin{vmatrix} 1 & 1 & 2x \\ x & 1+x & x^2 \\ x^2 & x & 1+x \end{vmatrix} + \begin{vmatrix} 1+x & x & x^2 \\ 1 & 1 & 2x \\ x^2 & x & 1+x \end{vmatrix} + \begin{vmatrix} 1+x & x & x^2 \\ x & 1+x & x^2 \\ 2x & 1 & 1 \end{vmatrix}$$

$= 5ax^4 + 4bx^3 + 3cx^2 + 2dx + \lambda$

Putting $x = 0$, we get

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \lambda$$

$\Rightarrow \quad 1 + 1 + 1 = \lambda$

$\Rightarrow \quad \lambda = 3$

20. The system of equations has infinite solutions if

$$\Delta = \begin{vmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{vmatrix} = 0$$

$\Rightarrow \quad 3(2\lambda+15) - 1(-\lambda-20) + 6(3-8) = 0$

$\Rightarrow \quad 6\lambda + 45 + \lambda + 20 - 30 = 0$

$\Rightarrow \quad 7\lambda = -35 \quad \therefore \quad \lambda = -5$

21. Applying $R_1 \to R_1 + R_3 - 2R_2$

$$\Delta = \begin{vmatrix} 0 & 0 & 0 & x+z-2y \\ 5 & 6 & 7 & y \\ 6 & 7 & 8 & z \\ x & y & z & 0 \end{vmatrix} = -(x+z-2y)\begin{vmatrix} 5 & 6 & 7 \\ 6 & 7 & 8 \\ x & y & z \end{vmatrix}$$

Again applying $C_1 + C_3 - 2C_2$, we get

$$\Delta = (x+z-2y)^2$$

22. Here $\Delta = 0$

$$\Rightarrow \begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0$$

[Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$\Rightarrow \quad (1+\sin^2\theta) + \cos^2\theta + 4\sin 4\theta = 0$

$\Rightarrow \quad 2 + 4\sin 4\theta = 0 \Rightarrow \sin 4\theta = -\frac{1}{2} = \sin\left(-\frac{\pi}{6}\right)$

$\therefore \quad 4\theta = n\pi + (-1)^n\left(-\frac{\pi}{6}\right)$

$\Rightarrow \quad \theta = \frac{n\pi}{4} + (-1)^{n+1}\frac{\pi}{24}$

When $n = 2, \theta = \frac{11\pi}{24}$

23. $$\Delta = \begin{vmatrix} 2a_1b_1 & a_1b_2 + a_2b_1 & a_1b_3 + a_3b_1 \\ a_1b_2 + a_2b_1 & 2a_2b_2 & a_2b_3 + a_3b_2 \\ a_1b_3 + a_3b_1 & a_3b_2 + a_2b_3 & 2a_3b_3 \end{vmatrix}$$

$$\therefore \quad \Delta = \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix}\begin{vmatrix} b_1 & a_1 & 0 \\ b_2 & a_2 & 0 \\ b_3 & a_3 & 0 \end{vmatrix} = 0$$

24. Applying $C_1 \to C_1 + C_2 + C_3$ and putting $1 + \omega + \omega^2 = 0$, we have

$$\Delta = \lambda\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & \lambda+\omega^2 & 1 \\ 1 & 1 & \lambda+\omega \end{vmatrix}$$

[Expanding and not making two zeros]

$= \lambda\{(\lambda^2 - \lambda) - 1(\lambda\omega) + 1(\omega - \lambda\omega^2 - \omega)\}$

$= \lambda[\lambda^2 - \lambda(1+\omega+\omega^2)] = \lambda^3$

25. $\Delta = x^n y^n z^n \begin{vmatrix} 1 & x^2 & x^4 \\ 1 & y^2 & y^4 \\ 1 & z^2 & z^4 \end{vmatrix}$

$= (xyz)^n (x^2 - y^2)(y^2 - z^2)(z^2 - x^2)$

$= \left(\frac{1}{y^2} - \frac{1}{x^2}\right)\left(\frac{1}{z^2} - \frac{1}{y^2}\right)\left(\frac{1}{x^2} - \frac{1}{z^2}\right)$ when $n = -4$

26. Since $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0,$

$\Rightarrow \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} (1 + xyz) = 0$

$\Rightarrow (x-y)(y-z)(z-x)(1+xyz) = 0$

$\Rightarrow xyz + 1 = 0$ $[\because x \neq y \neq z \text{ (given)}]$

$\therefore xyz = -1$

27. $f'(x) = -\begin{vmatrix} \sin(x+\alpha) & \sin(x+\beta) & \sin(x+\gamma) \\ \sin(x+\alpha) & \sin(x+\beta) & \sin(x+\gamma) \\ \sin(\beta-\alpha) & \sin(\gamma-\alpha) & \sin(\alpha-\beta) \end{vmatrix}$

$+ \begin{vmatrix} \cos(x+\alpha) & \cos(x+\beta) & \cos(x+\gamma) \\ \cos(x+\alpha) & \cos(x+\beta) & \cos(x+\gamma) \\ \sin(\beta-\gamma) & \sin(1-\alpha) & \sin(\alpha-\beta) \end{vmatrix}$

$= -0 + 0 = 0$

$\Rightarrow f(x)$ is a constant function $= c$ (say)

$\therefore f(\theta) - 2f(\theta) + f(\theta) = c - 2c + c = 0$

28. $\begin{vmatrix} a & x & p \\ b & y & q \\ c & z & r \end{vmatrix} = \lambda$, then

interchanging $C_3 \leftrightarrow C_2$ and according to properties of determinants, we have

$\therefore \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = -\begin{vmatrix} a & x & p \\ b & y & q \\ c & z & r \end{vmatrix} = -\lambda$

29. Operating $C_1 \to C_1 + C_2 + C_3$, we get $\Delta = 0$

30. Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ (2^x + 2^{-x})^2 & (3^x + 3^{-x})^2 & (5^x + 5^{-x})^2 \\ (2^x - 2^{-x})^2 & (3^x - 3^{-x})^2 & (5^x - 5^{-x})^2 \end{vmatrix}$

Applying $R_2 \to R_2 - R_3$, we get

$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 4 \cdot 2^x \cdot 2^{-x} & 4 \cdot 3^x \cdot 3^{-x} & 4 \cdot 5^x \cdot 5^x \\ (2^x - 2^{-x})^2 & (3^x - 3^{-x})^2 & (5^x - 5^{-x})^2 \end{vmatrix}$

$\Rightarrow \Delta = 4 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ (2^x - 2^{-x})^2 & (3^x - 3^{-x})^2 & (5^x - 5^{-x})^2 \end{vmatrix}$

$\therefore \Delta = 0$

31. Applying $C_1 \to C_1 - C_3$, and taking out $x - 4$ and expanding, we have

$$\Delta = (x-4)(x^2 + 4x - 25) = 0$$

The non-integral roots are given by the second factor and their sum S is -4.

32. Applying $C_1 \to C_1 - C_2$, we get

$$\Delta = \begin{vmatrix} 4 & (\sin\theta - \operatorname{cosec}\theta)^2 & 1 \\ 4 & (\cos\theta - \sec\theta)^2 & 1 \\ 4 & (\tan\theta - \cot\theta)^2 & 1 \end{vmatrix} = 0$$

33. L.H.S. $= a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$

$= a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix}$

(Applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + R_1$)

L.H.S. $= 4a^2b^2c^2 = \alpha a^2b^2c^2$

$\therefore \alpha = 4$

34. Multiplying R_1, R_2, R_3 by a, b, c respectively and hence dividing Δ_1 by abc, and then taking out a, b, c common from C_1, C_2, C_3 respectively, we get $\Delta_1 = \Delta_2$.

Now applying $R_1 \to R_1 - (R_2 + R_3)$ on Δ_2, we get

$$\Delta_2 = \begin{vmatrix} 0 & -2c^2 & -2b^2 \\ b^2 & c^2 + a^2 & b^2 \\ c^2 & c^2 & a^2 + b^2 \end{vmatrix}$$

Expanding with R_1

$\Delta_2 = 2c^2b^2(a^2 + b^2 - c^2) - 2b^2c^2(b^2 - c^2 - a^2)$

$= 2b^2c^2(2a^2) = 4a^2b^2c^2$

$\therefore \lambda = 4$

35. Expanding the determinant, we have

$$(2-y)(26-15y+y^2)-2(20-2y-18) + 3(8-15+3y)=0$$

$\Rightarrow$ $-y^3 + 17y^2 - 43y + 27 = 0$ or $y^3 - 17y^2 - 43y - 27 = 0$

$\therefore$ We have $(y-1)(y^2 - 16y + 27) = 0$

$\therefore$ Either $y = 1$ or $y^2 - 16y + 27 = 0$

$$\Rightarrow \quad y = \frac{16 \pm \sqrt{256-108}}{2} = \frac{8 \pm \sqrt{148}}{2}$$

which are not integers

Hence, the only integral root is $y = 1$.

36. Since x, y, z are in A.P., therefore, $x + z - 2y = 0$.

Now operating $R_1 \to R_1 + R_3 - 2R_2$, we get

$$\Delta = \begin{bmatrix} 0 & 0 & 2(x+z-2y) \\ a+3 & a+4 & a+2y \\ a+4 & a+5 & a+2z \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ a+3 & a+4 & a+2y \\ a+4 & a+5 & a+2z \end{bmatrix} = 0$$

37. Dividing C_1, C_2, C_3 by a, b, c respectively and then multiplying Δ by abc, we get

$$\Delta = abc \begin{vmatrix} \frac{1}{a}+1 & \frac{1}{b} & \frac{1}{c} \\ \frac{1}{a} & \frac{1}{b}+1 & \frac{1}{c} \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c}+1 \end{vmatrix}$$

Now applying $C_1 + C_2 + C_3$ and taking $1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ common and then making two zeros, we have

$$\therefore \quad \Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = abc\left(1+\sum\frac{1}{a}\right)$$

38. Applying $C_1 \to C_1 \to C_1 + C_3 - 2\cos x\, C_2$, we get

$$\Delta = \begin{vmatrix} a^2-2a\cos x+1 & a & 1 \\ 0 & \cos(n+1)x & \cos(n+2)x \\ 0 & \sin(n+1)x & \sin(n+2)x \end{vmatrix}$$

$[\because \cos nx + \cos(n+2)x = 2\cos(n+1)x\cos x$ and $\sin nx + \sin(n+2)x = 2\sin(n+1)x\cos x]$

Expanding along C_1, we get

$$\Delta = (a^2-2a\cos x+1)[\cos(n+1)x\sin(n+2)x - \sin(n+1)x\cos(n+2)x]$$

$$= (a^2-2a\cos x+1)\sin[(n+2)x-(n+1)x]$$

$$= (a^2-2a\cos x+1)\sin x$$

which is independent of n.

39. $$\Delta = \frac{1}{abc}\begin{vmatrix} ab^2c^2 & abc & ab+ac \\ a^2bc^2 & abc & bc+ab \\ a^2b^2c & abc & ac+bc \end{vmatrix}$$

$$= \frac{a^2b^2c^2}{abc}\begin{vmatrix} bc & 1 & ab+ac \\ ac & 1 & bc+ab \\ ab & 1 & ac+bc \end{vmatrix}$$

[Applying $C_1 \to C_1 + C_3$]

$$= abc\begin{vmatrix} ab+bc+ca & 1 & ab+ac \\ ab+bc+ca & 1 & bc+ab \\ ab+bc+ca & 1 & ac+bc \end{vmatrix}$$

$$= abc(ab+bc+ca)\begin{vmatrix} 1 & 1 & ab+ac \\ 1 & 1 & bc+ab \\ 1 & 1 & ac+bc \end{vmatrix}$$

$= 0$

40. Dividing C_1, C_2, C_3 by x, y, z respectively, we get

$$\Delta = xyz\begin{vmatrix} \frac{p}{x} & \frac{q}{y}-1 & \frac{r}{z}-1 \\ \frac{p}{x}-1 & \frac{q}{y} & \frac{r}{z}-1 \\ \frac{p}{x}-1 & \frac{q}{y}-1 & \frac{r}{z} \end{vmatrix} = 0$$

Applying $C_1 \to C_1 + C_2 + C_3$, taking $\left(\Sigma\frac{p}{x}-2\right)$ common, then making two zeros and expanding, we get

$$\therefore \quad \Delta = xyz\left(\Sigma\frac{p}{x}-2\right) = 0 \;\therefore \Sigma\frac{p}{x} = 2$$

41. Using the sum property, we get

$$\sum_{r=0}^{m}\Delta_r = \begin{vmatrix} \sum_{r=0}^{m}(2r-1) & \sum_{r=0}^{m}{}^mC_r & \sum_{r=0}^{m}1 \\ m^2-1 & 2^m & m+1 \\ \sin^2(m^2) & \sin^2(m) & \sin^2(m+1) \end{vmatrix}$$

But $\sum_{r=0}^{m}(2r-1) = \frac{1}{2}(m+1)(2m-1-1) = m^2-1$

$\sum_{r=0}^{m}{}^mC_r = 2^m$ and $\sum_{r=0}^{m}1 = m+1$. Therefore

$$\sum_{r=0}^{m}\Delta_r = \begin{vmatrix} m^2-1 & 2^m & m+1 \\ m^2-1 & 2^m & m+1 \\ \sin^2(m^2) & \sin^2(m) & \sin^2(m+1) \end{vmatrix} = 0$$

42. Since a, b and c are the roots of $x^3 + px + q = 0$,

$\therefore \quad a+b+c=0$

$$\therefore \quad \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$= 0$

43. Applying $R_1 \to R_1 + R_3 - 2R_2$ and taking 2 common from R_1, we get

$$\Delta = 2\begin{vmatrix} 1 & 1 & 1 \\ (x-1)^2 & x^2 & (x+1)^2 \\ x^2 & (x+1)^2 & (x+2)^2 \end{vmatrix}$$

Then making two zeros by $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$ and expanding, we get

$$2\begin{vmatrix} 1 & 0 & 0 \\ (x-1)^2 & 2x-1 & 4x \\ x^2 & 2x+1 & 4x+4 \end{vmatrix} = 2[(2x-1)(4x+4) - 4x(2x+1)]$$

$$= 2(-4) = -8$$

44. Explanation: $A = \begin{vmatrix} 0 & 2 & 1 \\ -2 & 0 & 3 \\ -1 & -3 & 0 \end{vmatrix}$

$A = 0(0+9) - 2(0+3) + 1(6-0)$

$A = 0$

The odd order skew-symmetric determinant is always equal to zero. The even order skew-symmetric determinant is always equal to a perfect square.

So, Assertion is true but Reason is not true.

45. Applying $C_1 \to C_1 + C_2 + C_3$ and taking common from C_1, we get

$$1+\omega^n+\omega^{2n}\begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ 1 & 1 & \omega^n \\ 1 & \omega^{2n} & 1 \end{vmatrix}$$

$1+\omega^n+\omega^{2n} = 0 \qquad [\because n \in N]$

Both Assertion and Reason are true and Assertion follows from Reason.

46. The minor of 5 in the determinant is

$$5 = M_{12} = \begin{vmatrix} 0 & 7 \\ -1 & 6 \end{vmatrix} = 0+7=7$$

So, Assertion is not true but Reason is true.

47. $D = \begin{vmatrix} a & a^3 & a^4-1 \\ b & b^3 & b^4-1 \\ c & c^3 & c^4-1 \end{vmatrix}$

$$= \begin{vmatrix} a & a^3 & a^4 \\ b & b^3 & b^4 \\ c & c^3 & c^4 \end{vmatrix} + \begin{vmatrix} a & a^3 & -1 \\ b & b^3 & -1 \\ c & c^3 & -1 \end{vmatrix}$$

$$= abc\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} - \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = 0$$

$D \Rightarrow abc\,(D_1) - (D_2) = 0 \qquad$ (i)

Now, $\quad D_1 = abc\begin{vmatrix} \frac{1}{a} & a & a^2 \\ \frac{1}{b} & b & b^2 \\ \frac{1}{c} & c & c^2 \end{vmatrix} = \begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix}$

Applying operations, we get

$$D_1 = (ab+bc+ca)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Applying operations $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$D_1 = (ab+bc+ca)(a-b)(b-c)(c-a) \qquad$ (ii)

$$D_2 = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$$

Applying operations $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$D_2 = (a-b)(b-c)(c-a)(a+b+c) \qquad$ (iii)

From (i), (ii) and (iii), we get

$abc(ab+bc+ca) = (a+b+c)$ [$\because$ a, b, c are different]

So, both Assertion and Reason are true.

48. $D(x) = \lambda_1 f_1(x) - \lambda_2 f_2(x) + \lambda_3 f(x)$

where $\lambda_1 = (bf-ce); \lambda_2 = (af-cd); \lambda_3 = (ae-bd)$

Then,

$$\int D dx = \int \lambda_1 f_1(x)dx + \int \lambda_2 f_2(x)dx + \int \lambda_3 f_3(x)dx + k$$

$$= \begin{vmatrix} \int f_1(x)dx & \int f_2(x)dx & \int f_3(x)dx \\ a & b & c \\ d & e & f \end{vmatrix}$$

Assertion is true and follows from Statement 2.

49. If $\Delta = \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$,

then $\begin{vmatrix} ka & kb & kc \\ kx & ky & kz \\ kp & kq & kr \end{vmatrix} = k^3 \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = k^3\Delta$

Thus, both Assertion and Reason are true and Assertion follows from Reason.

50. Applying $R_1 \to R_1 + R_3$, we get

$$\Delta = \begin{vmatrix} 1-i & -1 & \omega^2 - 1 \\ 1-i & -1 & \omega^2 - 1 \\ -i & -1+\omega-i & -1 \end{vmatrix} = 0$$

51. $x + iy = \begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix}$

$$= -3i \begin{vmatrix} 6i & 1 & 1 \\ 4 & -1 & -1 \\ 20 & i & i \end{vmatrix} = 0$$

$\Rightarrow$ $x = 0, y = 0$

52. If n is a multiple of 3, then all the rows are identical and $\Delta = 0$.
If n is not a multiple of 3, then $1 + \omega^n + \omega^{2n} = 0$.
Since $1 + \omega + \omega^2 = 0$,
applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 0 & \omega^n & \omega^{2n} \\ 0 & \omega^{2n} & 1 \\ 0 & 1 & \omega^n \end{vmatrix}$$

$\Rightarrow \Delta = 0$ for all integers n.

53. Operating $C_2 \to C_2 - 2C_3$, we get

for non-zero solution $\begin{vmatrix} 1 & 0 & a \\ 1 & b & b \\ 1 & 2c & c \end{vmatrix} = 0$

$\Rightarrow$ $2ac = ab + bc$ $\Rightarrow$ $b = \dfrac{2ab}{a+c}$

$\Rightarrow$ a, b, c are in H.P.

54. $a_n = a_1 r^{n-1}$, r is the common ratio.

$\log a_n = \log a_1 + (n-1)\log r$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$ in the given determinant, we get

$$\begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ 3\log r & 3\log r & 3\log r \\ 6\log r & 6\log r & 6\log r \end{vmatrix} = 0$$

Since R_2 and R_3 are proportional.

55. Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$f(x) = \begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 1 & 1+b^2x & (1+c^2)x \\ 1 & (1+b^2)x & 1+c^2x \end{vmatrix} \quad [\because\ a^2+b^2+c^2+2=0]$$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$= \begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 0 & 1-x & 0 \\ 0 & 0 & 1-x \end{vmatrix} = 1 - x^2$$

Hence the degree of $f(x) = 2$.

56. For no solution or infinitely many solutions,

$\begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = 0 \Rightarrow \alpha = 1, \alpha = -2.$

But for $\alpha = 1$, clearly there are infinitely many solutions and when we put $\alpha = -2$ in the given system of equations and adding them together, L.H.S. $\neq$ R.H.S., i.e. no solution.

57. Applying $R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$, we get

$$\begin{vmatrix} 0 & -x & 0 \\ 0 & x & -y \\ 1 & 1 & 1+y \end{vmatrix}$$

Expanding along R_1, $x(y) = xy$
$\therefore$ Divisible by both x and y.

58. For non-zero solution, $\begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$

$$1(1-a^2) + c(-c-ab) - b(ac+b) = 0$$
$$1 - a^2 - b^2 - c^2 - 2abc = 0$$
$$a^2 + b^2 + c^2 + 2abc = 1$$

59. As $\det(A) = \pm 1$, A^{-1} exists

and $A^{-1} = \dfrac{1}{\det(A)}(\text{adj}A) = \pm(\text{adj}A)$

All entries in adj(A) are integers.
$\therefore$ A^{-1} has integer entries.

60. $\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^n \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ a & -b & c \end{vmatrix}$

$= \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^n \begin{vmatrix} a+1 & a-1 & a \\ b+1 & b-1 & -b \\ c-1 & c+1 & c \end{vmatrix}$

$= \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^{n+1} \begin{vmatrix} a+1 & a & a-1 \\ b+1 & -b & b-1 \\ c-1 & c & c+1 \end{vmatrix}$

$(C_2 \leftrightarrow C_3)$

$= (1+(-1)^{n+2}) \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix}$

From above it is clear that n can be any odd integer.

61. Let

$$\Delta = \begin{vmatrix} x! & (x+1)x! & (x+2)(x+1)x! \\ (x+1)! & (x+2)(x+1)! & (x+3)(x+2)(x+1)! \\ (x+2)! & (x+3)(x+2)! & (x+4)(x+3)(x+2)! \end{vmatrix}$$

Taking common $x!$, $(x+1)!$ and $(x+2)!$ from R_1, R_2 and R_3 respectively

$$= x!(x+1)!(x+2)! \begin{vmatrix} 1 & (x+1) & (x+2)(x+1) \\ 1 & (x+2) & (x+3)(x+2) \\ 1 & (x+3) & (x+4)(x+3) \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_2$,

$$= x!(x+1)!(x+2)! \begin{vmatrix} 1 & (x+1) & (x+2)(x+1) \\ 0 & 1 & 2(x+2) \\ 0 & 1 & 2(x+3) \end{vmatrix}$$

$= x!(x+1)!(x+2)![1(2x+6-2x-4)]$

$= 2(x!)(x+1)!(x+2)!$

62. $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$,

$= \begin{vmatrix} 3 & 1 & 1 \\ 0 & -1-\omega^2 & \omega^2 \\ 0 & \omega^2 & \omega^4 \end{vmatrix} = 3(\omega^2 - \omega) = 3\omega(\omega - 1)$

63. If $C = 2\cos\theta$, expanding along R_1,

$\Delta = C(C^2-1) - 1(C-6) + 0(1-6C)$

$= C^3 - 2C + 6$

$= 8\cos^3\theta - 4\cos\theta + 6$

64. $\sum_{r=1}^{n} D_r = \begin{vmatrix} \sum_{r=1}^{n} 2^{r-1} & \sum_{r=1}^{n} 2\cdot 3^{r-1} & \sum_{r=1}^{n} 4\cdot 5^{r-1} \\ \alpha & \beta & \gamma \\ 2^n-1 & 3^n-1 & 5^n-1 \end{vmatrix}$

Now, $\sum_{r=1}^{n} 2^{r-1} = 1 + 2 + 2^2 + \cdots + 2^{n-1} = \frac{1\cdot(2^n-1)}{(2-1)}$

$= 2^n - 1$

$\sum_{r=1}^{n} 2\cdot 3^{r-1} = 2(1+3+3^2+\cdots+3^{n-1}) = \frac{2(3^n-1)}{(3-1)}$

$= 3^n - 1$

$\sum_{r=1}^{n} 4\cdot 5^{r-1} = 4[1+5+5^2+\cdots+5^{n-1}] = \frac{4(5^n-1)}{(5-1)}$

$= 5^n - 1$

$\therefore \sum_{r=1}^{n} D_r = \begin{vmatrix} 2^n-1 & 3^n-1 & 5^n-1 \\ \alpha & \beta & \gamma \\ 2^n-1 & 3^n-1 & 5^n-1 \end{vmatrix} = 0$

65. Expanding along R_1, we get

$-12(30+1) - 0(0+2) + \lambda(0-4) = -360$

$\Rightarrow -372 - 4\lambda = -360 \Rightarrow -4\lambda = 12 \Rightarrow \lambda = -3$

66. $\Delta = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$

$= \begin{vmatrix} 0 & 0 & 2 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$ $(R_1 \to R_1 + R_3)$

$\Rightarrow \quad \Delta = 2(\sin^2\theta + 1)$ (i)

Now, $0 \le \sin^2\theta \le 1$

$\Rightarrow \quad 1 \le 1 + \sin^2\theta \le 2 \quad \Rightarrow \quad 2 \le 2(1+\sin^2\theta) \le 4$

$\Rightarrow \quad 2 \le \Delta \le 4$ [From Eq. (i)]

So, $\Delta \in [2, 4]$

67. Operating $R_1 \to R_1 - R_2 - R_3$, we get

$$\Delta = \begin{vmatrix} 0 & -2z & -2y \\ y & z+x & y \\ z & z & x+y \end{vmatrix}$$

Expanding, we get $\Delta = 4xyz$.

68. Statement 1: Determinant of a skew-symmetric matrix of odd order is zero.

Statement 2: $\det(A^T) = \det(A)$

$\det(-A) = (-1)^n \det(A)$, where A is an $n \times n$ order matrix.

MULTIPLE CHOICE TYPE QUESTIONS—LEVEL 1

1. If $a, b, c > 0$ and $x, y, z \in \mathbb{R}$, then the determinant

$$\begin{vmatrix} (a^x+a^{-x})^2 & (a^x+a^{-x})^2 & 1 \\ (b^y+a^{-y})^2 & (b^y-a^{-y})^2 & 1 \\ (c^z+c^{-z})^2 & (c^z-c^{-z})^2 & 1 \end{vmatrix} = ?$$

(a) $a^x b^y c^z$ (b) $a^{-x} b^{-y} c^{-z}$
(c) $a^{2x} b^{2y} c^{2z}$ (d) zero

2. If A, B and C are the angles of a triangle ABC, then the value of the determinant

$$\begin{vmatrix} \sin\frac{A}{2} & \sin\frac{B}{2} & \sin\frac{C}{2} \\ \sin(A+B+C) & \sin\frac{B}{2} & \cos\frac{A}{2} \\ \cos\frac{(A+B+C)}{2} & \tan(A+B+C) & \sin\frac{C}{2} \end{vmatrix}$$

is less than or equal to
(a) 1/2 (b) 1/4
(c) 1/8 (d) none of these

3. If $f(x) = \begin{vmatrix} x & \cos x & e^{x^2} \\ \sin x & x^2 & \sec x \\ \tan x & 1 & 2 \end{vmatrix}$, then the value of $\int_{-\pi/2}^{\pi/2} f(x)dx$ is equal to
(a) 0 (b) 1
(c) 2 (d) none of these

4. The value of the $\Delta = \begin{vmatrix} a^3-x & a^4-x & a^5-x \\ a^5-x & a^6-x & a^7-x \\ a^7-x & a^8-x & a^9-x \end{vmatrix}$ is
(a) 0 (b) $(a^3-1)(a^6-1)(a^9-1)$
(c) $(a^3+1)(a^6+1)(a^9+1)$ (d) $a^{15}-1$

5. The determinant $\Delta = \begin{vmatrix} a^2(1+x) & ab & ac \\ ab & b^2(1+x) & bc \\ ac & bc & c^2(1+x) \end{vmatrix}$ is divisible by
(a) $1+x$ (b) $(1+x)^2$
(c) x^2 (d) none of these

6. If $U_n = \begin{vmatrix} n & 1 & 5 \\ n^2 & 2N+1 & 2N+1 \\ n^3 & 3N^2 & 3N+1 \end{vmatrix}$, then $\sum_{n=1}^{N} U_n$ is equal to
(a) $2\sum_{n=1}^{N} n$ (b) $2\sum_{n=1}^{N} n^2$
(c) $\frac{1}{2}\sum_{n=1}^{N} n^2$ (d) 0

7. If $\begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 6x+2 & 9x+3 & 12x \\ 8x+1 & 12x & 16x+2 \end{vmatrix} = 0$, then x is equal to
(a) $x=0$ (b) $x=-11$
(c) $x=97$ (d) $x=-\frac{11}{97}$

8. The determinant $\Delta = \begin{vmatrix} b & c & b\alpha+c \\ c & d & c\alpha+d \\ b\alpha+c & c\alpha+d & a\alpha^3-c\alpha \end{vmatrix}$ is equal to zero if
(a) b, c, d are in A.P.
(b) b, c, d are in G.P.
(c) b, c, d are in H.P
(d) α is a root of $ax^3-bx^2-3cx-d=0$

9. $\begin{vmatrix} 103 & 115 & 114 \\ 111 & 108 & 106 \\ 104 & 113 & 116 \end{vmatrix} + \begin{vmatrix} 113 & 116 & 104 \\ 108 & 106 & 111 \\ 115 & 114 & 103 \end{vmatrix}$ is equal to
(a) a positive number (b) a negative number
(c) zero (d) none of these

10. $\Delta = \begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}$ is equal to
(a) abc (b) $a^2b^2c^2$
(c) $ab+bc+ca$ (d) none of these

11. $\Delta = \begin{vmatrix} x^2+y^2 & ax+by & px+qy \\ ax+by & a^2+b^2 & ap+bq \\ px+qy & ap+bq & p^2+q^2 \end{vmatrix}$ is equal to
(a) $p+q$ (b) $a+b+c$
(c) $x+y+z$ (d) 0

12. $\Delta = \begin{vmatrix} 1+a^2+a^4 & 1+ab+a^2b^2 & 1+ac+a^2c^2 \\ 1+ab+a^2b^2 & 1+b^2+b^4 & 1+bc+b^2c^2 \\ 1+ac+a^2c^2 & 1+bc+b^2c^2 & 1+c^2+c^4 \end{vmatrix}$ is equal to

(a) $(a+b+c)^6$ (b) $(a-b)^2(b-c)^2(c-a)^2$
(c) $4(a-b)(b-c)(c-a)$ (d) none of these

13. The determinant $\begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ 0 & xp+y & yp+z \end{vmatrix} = 0$ for all $p \in \mathbb{R}$ if
(a) x, y, z are in A.P. (b) x, y, z are in G.P.
(c) x, y, z are in H.P. (d) xy, yz, zx are in A.P.

14. If the determinant $\begin{vmatrix} a & a+d & a+2d \\ a^2 & (a+d)^2 & (a+2d)^2 \\ 2a+3d & 2(a+d) & 2a+d \end{vmatrix} = 0$, then
(a) $d = 0$ (b) $a + d = 0$
(c) $d = 0$ if $a + d = 0$ (d) none of these

15. If x, y, z are integers in A.P. lying between 1 and 9 and $x51$, $y41$ and $z31$ are three-digit numbers, then the value of $\begin{vmatrix} 5 & 4 & 3 \\ x51 & y41 & z31 \\ x & y & z \end{vmatrix}$ is
(a) $x+y+z$ (b) $x-y+z$
(c) 0 (d) none of these

16. The value of $\begin{vmatrix} a_1x+b_1y & a_2x+b_2y & a_3x+b_3y \\ b_1x+a_1y & b_2x+a_2y & b_3x+a_3y \\ b_1x+a_1 & b_2x+a_2 & b_3x+a_3 \end{vmatrix}$ is equal to
(a) x^2+y^2 (b) 0
(c) $a_1a_2a_3x^2+b_1b_2b_3y^2$ (d) none of these

17. If α and β are non-real numbers satisfying $x^3 - 1 = 0$, then the value of $\begin{vmatrix} \lambda+1 & \alpha & \beta \\ \alpha & \lambda+\beta & 1 \\ \beta & 1 & \lambda+\alpha \end{vmatrix}$ is equal to
(a) 0 (b) λ^3
(c) $\lambda^3 + 1$ (d) none of these

18. The value of $\begin{vmatrix} {}^{10}C_4 & {}^{10}C_5 & {}^{11}C_m \\ {}^{11}C_6 & {}^{11}C_7 & {}^{12}C_{m+2} \\ {}^{12}C_8 & {}^{12}C_9 & {}^{13}C_{m+4} \end{vmatrix}$ is equal to zero when m is
(a) 6 (b) 4
(c) 5 (d) none of these

MULTIPLE CHOICE TYPE QUESTIONS—LEVEL 2

1. $\begin{vmatrix} \cos C & \tan A & 0 \\ \sin B & 0 & -\tan A \\ 0 & \sin B & \cos C \end{vmatrix}$ has the value
(a) 0 (b) 1
(c) $\sin A \sin B \sin C$ (d) none of these

2. The value of $\begin{vmatrix} x & x^2-yz & 1 \\ y & y^2-zx & 1 \\ z & z^2-xy & 1 \end{vmatrix}$ is
(a) 1 (b) -1
(c) 0 (d) $-xyz$

3. If $\begin{vmatrix} x^k & x^{k+2} & x^{k+3} \\ y^k & y^{k+2} & y^{k+3} \\ z^k & z^{k+2} & z^{k+3} \end{vmatrix} = (x-y)(y-z)(z-x)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$, then
(a) $k = -3$ (b) $k = -1$
(c) $k = 1$ (d) $k = 3$

4. If $\sqrt{-1} = i$ and ω is a non-real cube root of unity, then the value of $\begin{vmatrix} 1 & \omega^2 & 1+i+\omega^2 \\ -i & -1 & -1-i+\omega \\ 1-i & \omega^2-1 & -1 \end{vmatrix}$ is equal to
(a) 1 (b) i
(c) ω (d) 0

5. If the determinant $\begin{vmatrix} a+p & l+x & u+f \\ b+q & m+y & v+g \\ c+r & n+z & w+h \end{vmatrix}$ splits into exactly K determinants of order 3, each element of which contains only one term, then the value of K is
(a) 6 (b) 7
(c) 8 (d) 9

6. The value of $\begin{vmatrix} i^m & i^{m+1} & i^{m+2} \\ i^{m+5} & i^{m+4} & i^{m+3} \\ i^{m+6} & i^{m+7} & i^{m+8} \end{vmatrix}$, where $i = \sqrt{-1}$, is
(a) 1 if m is a multiple of 4
(d) 0 for all real m
(c) $-i$ if m is a multiple of 3
(d) none of these

7. Let $\begin{vmatrix} \lambda^2+3\lambda & \lambda-1 & \lambda+3 \\ \lambda+1 & -2\lambda & \lambda-4 \\ \lambda-1 & \lambda+4 & 3\lambda \end{vmatrix} = p\lambda^4 + q\lambda^3 + r\lambda^2 + s\lambda + t$ be an identity in λ, where $p, q, r\ s, t$, are independent of λ, then the value of t is

(a) 4 (b) 0
(c) 1 (d) none of these

8. If n is not a multiple of 3 and 1, ω, ω^2 are the cube roots of unity, then $\Delta = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^{2n} & 1 & \omega^n \\ \omega^n & \omega^{2n} & 1 \end{vmatrix}$ is equal to

(a) 0 (b) ω
(c) ω^2 (d) 1

9. If $\Delta(x) = \begin{vmatrix} 1 & \cos x & 1-\cos x \\ 1+\sin x & \cos x & 1+\sin x-\cos x \\ \sin x & \sin x & 1 \end{vmatrix}$, then $\int_0^{\pi/2} \Delta(x)dx$ is equal to

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$
(c) 0 (d) $-\frac{1}{2}$

10. If a, b, c are real numbers, and
$D = \begin{vmatrix} a & 1+2i & 3-5i \\ 1-2i & b & -7-3i \\ 3+5i & -7+3i & c \end{vmatrix}$, then D is

(a) purely real (b) purely imaginary
(c) non-real (d) an integer

11. If $i = \sqrt{-1}$, $\sqrt[4]{1} = \alpha, \beta, \gamma, \delta$ then $\begin{vmatrix} \alpha & \beta & \gamma & \delta \\ \beta & \gamma & \delta & \alpha \\ \gamma & \delta & \alpha & \beta \\ \delta & \alpha & \beta & \gamma \end{vmatrix}$ is equal to

(a) i (b) $-i$
(c) 1 (d) 0

12. Let $\Delta = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$, then

(a) $x + a$ is a factor of Δ
(b) $(x + a)^2$ is a factor of Δ
(c) $(x + a)^3$ is a factor of Δ
(d) $(x + a)^4$ is not a factor of Δ

13. The system of equations $ax + 4y + z = 0$, $bx + 3y + z = 0$ and $cx + 2y + z = 0$ has non-trivial solutions if a, b, c are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

14. The system of equations

$$2x - y + z = 0$$
$$x - 2y + z = 0$$
$$\lambda x - y + 2z = 0$$

has an infinite number of non-trivial solutions for

(a) $\lambda = 1$ (b) $\lambda = 5$
(c) 5 (d) no real value of λ

ANSWERS

Level 1

1. (d) **2.** (c) **3.** (a) **4.** (a) **5.** (c) **6.** (b) **7.** (d) **8.** (b) and (d) **9.** (c)
10. (d) **11.** (d) **12.** (b) **13.** (b) **14.** (c) **15.** (c) **16.** (b) **17.** (b) **18.** (c)

Level 2

1. (a) **2.** (c) **3.** (b) **4.** (d) **5.** (c) **6.** (b) **7.** (b) **8.** (a) **9.** (d) **10.** (a)
11. (d) **12.** (a), (b) and (d) **13.** (a) **14.** (c)

-: THE END :-

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