ST. LAWRENCE HIGH SCHOOL

A JESUIT CHRISTIAN MINORITY INSTITUTION

STUDY MATERIAL-4

SUBJECT – MATHEMATICS

Pre-test

Chapter: MATRICES AND DETERMINANTS **Class: XII**

Topic: DETERMINANTS **Date: 12.05.2020**

PART 1

Determinants

DEFINITION OF DETERMINANT

A determinant is a pure number associated with a square matrix. Corresponding to each square matrix A, there is associated an

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

expression, called the determinant of A, denoted by det A or $|A|$ and written as

$$|A| = \det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

A matrix is an arrangement of numbers and it has no fixed value, but a determinant is a number and it has a fixed value. A determinant having n rows and n columns is called a **determinant of order n**.

Note

- An arrangement of numbers is known as matrix and a matrix having equal number of rows and columns is known as a square matrix.
- The determinant of a matrix, which is not a square matrix, cannot be found out.
- A matrix has no definite value while each determinant has a defined value.

DETERMINANT OF A SQUARE MATRIX OF ORDER 1

Let $A = [a_{11}]$ be a 1×1 matrix, then the determinant of A is the number a_{11} itself, i.e. $|a_{11}| = a_{11}$.

DETERMINANT OF A SQUARE MATRIX OF ORDER 2

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a 2×2 matrix, then

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

i.e. the determinant of a 2×2 matrix is obtained by taking the product of the entries on the main diagonal and subtracting from it the product of the entries on the other diagonal.

DETERMINANT OF A SQUARE MATRIX OF ORDER 3

The arrangement of 9 elements in 3 rows and 3 columns represents a determinant of order 3, such that

Columns $C_1 \; C_2 \; C_3$, Rows R_1, R_2, R_3

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \Delta$$

The representation is usually shown as Δ and the value of the determinant is

$$\Delta = a_{11}\underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} - a_{12}\underbrace{\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}} + a_{13}\underbrace{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}$$

determinant left after removing 1st row and 1st column; determinant left after removing 1st row and 2nd column; determinant left after removing 1st row and 3rd column

The above is the expansion of the determinant along the first row.

Example: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, then

$$= |A| = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix}$$

$$= 1\,(2-1) - 2(3-1) + 3(3-2)$$

$$= 1 - 2(2) + 3 = 4 - 4 = 0.$$

MINORS AND COFACTORS

1. Minors

Let $A = [a_{ij}]$ be a square matrix of order n. Then the minor M_{ij} of a_{ij} in A is the determinant of a square sub-matrix of order $(n-1)$ obtained by removing the ith row and jth column of A.

Example: If $A = \begin{vmatrix} 1 & 2 & 3 \\ -3 & 2 & -1 \\ 2 & -4 & 3 \end{vmatrix}$, then

$$M_{11} = \begin{vmatrix} 2 & -1 \\ -4 & 3 \end{vmatrix} = 6 - 4 = 2,$$

$$M_{12} = \begin{vmatrix} -3 & -1 \\ 2 & 3 \end{vmatrix} = -9 + 2 = -7$$

$$M_{13} = \begin{vmatrix} -3 & 2 \\ 2 & -4 \end{vmatrix} = 12 - 4 = 8, \; M_{21} = \begin{vmatrix} 2 & 3 \\ -4 & 3 \end{vmatrix} = 6 + 12 = 18$$

$$M_{22} = \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = 3 - 6 = -3, \; M_{23} = \begin{vmatrix} 1 & 2 \\ 2 & -4 \end{vmatrix} = -4 - 4 = -8$$

$$M_{31} = \begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix} = -2 - 6 = -8,$$

$$M_{32} = \begin{vmatrix} 1 & 3 \\ -3 & -1 \end{vmatrix} = -1 + 9 = 8$$

and $M_{33} = \begin{vmatrix} 1 & 2 \\ -3 & 2 \end{vmatrix} = 2 + 6 = 8.$

2. Cofactors

Let $A = [a_{ij}]$ be a square matrix of order n. Then the cofactor C_{ij} of a_{ij} in A is $(-1)^{i+j}$ times M_{ij}, where M_{ij} is the minor of a_{ij} in A.

$\therefore \qquad C_{ij} = (-1)^{i+j} M_{ij}$

Example: If $A = \begin{vmatrix} 4 & -7 \\ -3 & 2 \end{vmatrix}$, then the cofactors are

$$C_{11} = (-1)^{1+1} M_{11} = M_{11} = 2$$

$$C_{12} = (-1)^{1+2} M_{12} = -M_{12} = -(-3) = 3$$

$$C_{21} = (-1)^{2+1} M_{21} = -M_{21} = 7$$

and $\qquad C_{22} = (-1)^{2+2} M_{22} = M_{22} = 4.$

SARRUS RULE FOR EXPANSION

***Method I:** Using the Sarrus rule*

Sarrus gave a rule for a determinant of order 3. The following diagram, called Sarrus diagram, enables us to write the value of a determinant of order 3 in a very convenient manner.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - c_3a_2b_1 - b_3c_2a_1$$

Let $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ be a determinant of order 3. Write the elements as shown in Fig. 5.1. Multiply the elements joined by arrows. Assign the positive sign to an expression if it is formed by a downward arrow and the negative sign to an expression if it is formed by an upward arrow. Note that the first two columns written to the right of the determinant complete the process. The value of the determinant is given by $a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - c_3a_2b_1 - b_3c_2a_1$.

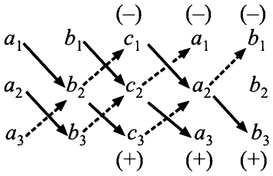
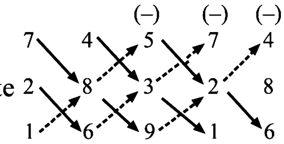


Fig. 5.1

Illustration 1. Find the value of the determinant

$A = \begin{vmatrix} 7 & 4 & 5 \\ 2 & 8 & 3 \\ 1 & 6 & 9 \end{vmatrix}$, using the Sarrus rule.

Solution: We write



$$= (7\times8\times9) + (4\times3\times1) + (5\times2\times6) - (1\times8\times5)$$
$$-(6\times3\times7) - (9\times2\times4)$$
$$= 504 + 12 + 60 - 40 - 126 - 72 = 576 - 238 = 338.$$

Method II: *Using cofactors*

The value of a determinant can also be evaluated by multiplying the elements of any one row (or column) by its corresponding cofactors and then summing the resulting products.

Example: Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then according to definition,

$$\Delta = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\Rightarrow \quad \Delta = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Also, $\Delta = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$

$\Delta = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$

and $\Delta = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$ and similarly the other results can be obtained.

Note Evaluation of a determinant using the cofactor rule is much more efficient than using the Sarrus rule for higher-order determinants.

PROPERTIES OF DETERMINANTS

Determinants have some properties that are useful as they permit to generate equal determinants with different and simpler configurations of entries. This, in turn, helps to find values of determinants with efficiency. The properties of determinants are as follows:

Property 1: The value of a determinant is not changed when its rows are changed into corresponding columns. Naturally when rows are changed into corresponding columns, then columns will be changed into corresponding rows. This operation is called taking the transpose of a determinant.

Example: For any determinant, we have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$

Property 2: If two adjacent rows or columns of a determinant are interchanged, the value of the determinant so obtained is the negative of the value of the original determinant.

Example: For any determinant, we have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Note If n interchanges of rows or columns take place, then the resulting determinant is $(-1)^n$ times the given determinant.

Property 3: The value of a determinant is zero if any two rows or columns are identical.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \quad (\text{here } R_1 = R_2)$$

Property 4: The value of a diagonal matrix, upper triangular matrix and lower triangular matrix is equal to the product of the principal diagonal elements.

Example: (i) Diagonal matrix

Let $\Delta = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix}$. Expanding along R_1 (row 1),

we have, $\Delta = a_{11}(a_{22}a_{33} - 0) - 0 + 0$

$\Rightarrow \quad \Delta = a_{11}a_{22}a_{33}$ = Product of principal diagonal elements

(ii) Upper triangular matrix

Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix}$. Expanding along C_1 (column 1), we have, $\Delta = a_{11}(a_{22}a_{33} - a_{23}\cdot 0) - 0 + 0$

$\Rightarrow \quad \Delta = a_{11}a_{22}a_{33}$ = Product of principal diagonal elements

(iii) Lower triangular matrix

Let $\Delta = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$. Expanding along R_1, we have

$$\Delta = a_{11}(a_{22}a_{33} - a_{32}\cdot 0) - 0 + 0$$

$\Rightarrow \quad \Delta = a_{11}a_{22}a_{33}$ = Product of principal diagonal elements

Property 5: A common factor of all elements of any row (or any column) may be taken outside the sign of the determinant. In other words, if all the elements of the same row (or the same column) are multiplied by a certain number, then the determinant becomes multiplied by that number.

Example: For any determinant, we have

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Property 6: If every element of some column (or row) is the sum of two terms, then the given determinant is equal to the addition of two determinants—the first one with the first term of the sum and the second one with the second term of the sum. The remaining elements of both the determinants are the same as in the given determinant.

$$\begin{vmatrix} a_{11}+\alpha_1 & a_{12} & a_{13} \\ a_{21}+\alpha_2 & a_{22} & a_{23} \\ a_{31}+\alpha_3 & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} \alpha_1 & a_{12} & a_{13} \\ \alpha_2 & a_{22} & a_{23} \\ \alpha_3 & a_{32} & a_{33} \end{vmatrix}$$

Property 7: If the elements of a row (or column) of a determinant are added k times to the corresponding elements of another row (or column), then the value of the determinant thus obtained equals the value of the original determinant.

Example: Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ be any determinant.

Then $$\begin{vmatrix} a_{11}+ka_{12} & a_{12} & a_{13} \\ a_{21}+ka_{22} & a_{22} & a_{23} \\ a_{31}+ka_{32} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

[Applying $C_1 \to C_1 + kC_2$]

or $$\begin{vmatrix} a_{11}+ka_{12}+ma_{13} & a_{12} & a_{13} \\ a_{21}+ka_{22}+ma_{23} & a_{22} & a_{23} \\ a_{31}+ka_{32}+ma_{33} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

[Applying $C_1 \to C_1 + kC_2 + mC_3$]

Property 8: If each element of a row (or column) of a determinant is zero, then the value of the determinant is zero.

Property 9: If each element of a determinant is a function of x, such that

$$\Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$$

and if at $x = \alpha$, $\Delta(x) = 0$, i.e. $\Delta(\alpha) = 0$,

then $x = \alpha$ is called the root of the determinant function and conversely $(x - \alpha)$ is the factor of the determinant.

Note If r rows (or columns) become identical when a is substituted for x, then $(x - a)^{r-1}$ is the factor of the given determinant.

☞ PRODUCT OF DETERMINANTS

Let $\Delta_1 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix}$ be any two determinants of third order, then

$$\Delta_1 \cdot \Delta_2 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}\begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix}$$

$$\Rightarrow \quad \Delta_1 \cdot \Delta_2 = \begin{vmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} & a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33} \end{vmatrix}$$

i.e. the (i, j)th element in $(\Delta_1 \cdot \Delta_2)$ is obtained by adding the products of the corresponding elements of the ith row of Δ_1 and the jth column of Δ_2.

Note Since we know that the value of the determinant does not change by interchanging the rows and columns, therefore, while writing down the product of two determinants of the same order, we can also follow the row by row multiplication rule, rather than only the row by column rule.

Also column by column and column by row methods of operation on product of determinants will yield the same result.

☞ DIFFERENTIATION OF A DETERMINANT

Differentiation of a determinant is equal to the sum of differentials distributed on each row (or column) one at a time.

Thus, if Δ is a determinant of order 3, such that

$$\Delta(x)=\begin{vmatrix} f(x) & g(x) & h(x) \\ p(x) & q(x) & r(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$$

where $f, g, h, p, q, r, u, v, w$ are functions of x alone. Then

$$\frac{d\Delta(x)}{dx}=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ p(x) & q(x) & r(x) \\ u(x) & v(x) & w(x) \end{vmatrix}+\begin{vmatrix} f(x) & g(x) & h(x) \\ p'(x) & q'(x) & r'(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$$
$$+\begin{vmatrix} f(x) & g(x) & h(x) \\ p(x) & q(x) & r(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}.$$

Illustration 2. If a, b, c are in A.P. and

$$f(x)=\begin{vmatrix} x+a & x^2+1 & 1 \\ x+b & 2x^2-1 & 1 \\ x+c & 3x^2-2 & 1 \end{vmatrix}, \text{ then find } f'(x).$$

Solution: $f(x)=\begin{vmatrix} x+a & x^2+1 & 1 \\ x+b & 2x^2-1 & 1 \\ x+c & 3x^2-2 & 1 \end{vmatrix}$

Differentiating column by column,

$$f'(x)=\begin{vmatrix} 1 & x^2+1 & 1 \\ 1 & 2x^2-1 & 1 \\ 1 & 3x^2-2 & 1 \end{vmatrix}+\begin{vmatrix} x+a & 2x & 1 \\ x+b & 4x & 1 \\ x+c & 6x & 1 \end{vmatrix}$$
$$+\begin{vmatrix} x+a & x^2+1 & 0 \\ x+b & 2x^2-1 & 0 \\ x+c & 3x^2-2 & 0 \end{vmatrix}$$

0 ($\because C_1$ and C_3 are identical)

$$+2x\begin{vmatrix} x+a & 1 & 1 \\ x+b & 2 & 1 \\ c-b & 1 & 0 \end{vmatrix} (R_3 \to R_3 - R_2)$$

$$=2x\begin{vmatrix} x+a & 1 & 1 \\ b-a & 1 & 0 \\ c-b & 1 & 0 \end{vmatrix} (R_2 \to R_2 - R_1)$$

$$= 2x(2b-a-c)$$

$$= 2 \times 0 = 0 \quad \left(\because a,b,c \text{ are in A.P.} \Rightarrow b=\frac{a+c}{2}\right)$$

☞ INTEGRATION OF A DETERMINANT

Let $\Delta(x)=\begin{vmatrix} a_{11}(x) & a_{12}(x) & a_{13}(x) \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$\therefore \quad \int \Delta(x)dx=\begin{vmatrix} \int a_{11}(x)dx & \int a_{12}(x)dx & \int a_{13}(x)dx \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Illustration 3. If $f(x)=\begin{vmatrix} x^3 & \sin x \\ 5 & 7 \end{vmatrix}$, then find the value of $\int_{-a}^{a} f(x)dx$.

Solution: $\int_{-a}^{a} f(x)dx=\begin{vmatrix} \int_{-a}^{a} x^3dx & \int_{-a}^{a} \sin x dx \\ 5 & 7 \end{vmatrix}=\begin{vmatrix} 0 & 0 \\ 5 & 7 \end{vmatrix}=0$

Note The above formula is applicable only if there is a variable only in one row or column, otherwise expand the determinant and then integrate.

☞ APPLICATIONS OF DETERMINANTS IN GEOMETRY

1. Area of a triangle: If $(x_1\, y_1)$, $(x_2\, y_2)$ and (x_3y_3) are the vertices of a triangle, then

the area of the triangle = $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$$=\frac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]$$

Illustration 4. Using determinants, find the area of ΔABC whose vertices A, B and C are the points (2, 3), (5, 4) and (–1, 6) respectively.

Solution: Area of $\Delta ABC=\frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ 5 & 4 & 1 \\ -1 & 6 & 1 \end{vmatrix}$

$$=\frac{1}{2}[2(4-6)-5(3-6)-1(3-4)]$$

$$=\frac{1}{2}[2(-2)-5(-3)-1(-1)]$$

$$=\frac{1}{2}[-4+15+1]=6 \text{ sq. units.}$$

2. Condition of collinearity of three points: Let the three points be $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then these points will be collinear if

the area of $\Delta ABC=0$

$$\Rightarrow \quad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0$$

Illustration 5. Show that the points $A(1, -1)$ $B(2, 1)$ and $C(x_3, y_3)$, are collinear.

Solution: We know that three points are collinear if the area of the triangle formed by them is zero.

Now the area of ΔABC with vertices $A(1,-1)$, $B(2, 1)$ and $C(4, 5)$ is given by

$$\frac{1}{2}\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ 4 & 5 & 1 \end{vmatrix}$$

$$= \frac{1}{2}[1(1-5)+1(2-4)+1(10-4)]$$

$$= \frac{1}{2}[-4-2+6]=0$$

Hence, the points $(1, -1)$, $(2, 1)$ and $(4, 5)$ are collinear.

3. Equation of a straight line passing through two points: Let the the two points be $A(x_1, y_1)$ and $B(x_2, y_2)$ and $P(x, y)$be a point on the line joining points A and B, then the equation of the line is given by

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Illustration 6. Using determinants, find the equation of the line joining the points (1, 2) and (3, 6).

Solution: Let $P(x, y)$ be any point on the line AB.

Then the area $(\Delta ABP) = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ x & y & 1 \end{vmatrix}$

$$= \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ x-1 & y-2 & 0 \end{vmatrix} \quad \left[\text{Applying } \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}\right]$$

$$= \frac{1}{2}[2(y-2)-4(x-1)]$$

$$= y-2-2x+2 = y-2x$$

As the points are collinear, the area of $\Delta ABP = 0$.

$\therefore$ $y - 2x = 0$, which is the required equation of the line AB.

SOLVING SYSTEM OF LINEAR EQUATIONS

Let n simultaneous equations in n unknowns $x_1, x_2, x_3, \ldots, x_n$ be

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= d_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{1n}x_n &= d_2 \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad\quad\quad \vdots \quad & \\ a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots + a_{in}x_n &= d_i \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad\quad\quad \vdots \quad & \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= d_n \end{aligned}$$

The above system of equations is said to be a system of homogeneous equation if

$d_1 = d_2 = d_3 = \cdots = d_i = \cdots = d_n = 0$. But if at least one of d_1, d_2, d_3 …, d_i, …, d_n is non-zero, the system of equations is said to form a non-homogeneous system.

SOME IMPORTANT TERMS DEFINED

Consistent system: If the system of equations has an infinite number of solutions, the system is said to be a *consistent system* or *indeterminant system* of equations.

Inconsistent system: If the system of equations has no solution, the system of equations is said to be an inconsistent system of equations.

CRAMMER'S RULE

Consider the following three linear simultaneous equations in x, y, z:

$$a_1x + b_1y + c_1z = d_1 \quad \text{(i)}$$

$$a_2x + b_2y + c_2z = d_2 \quad \text{(ii)}$$

$$a_3x + b_3y + c_3z = d_3 \quad \text{(iii)}$$

If $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$

$$\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

then using the Crammer's rule of determinants,

$$\frac{x}{\Delta_1} = \frac{y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{1}{\Delta}, \text{ i.e. } x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}$$

The above can be shown diagrammatically as in Figs. 5.2 and 5.3.

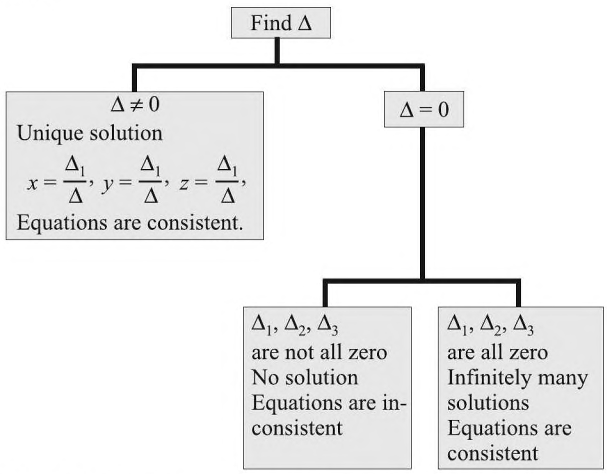


Fig. 5.2 Solution of a non-homogeneous system of linear equations.

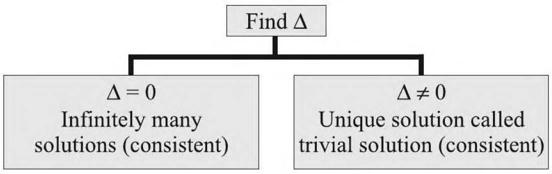


Fig. 5.3 Solution of a homogeneous system of linear equations.

Illustration 7. The system of equations $x + y + z = 2,\ 2x + y - z = 3$ and $3x + 2y + kz = 4$ has a unique solution if

(a) $k = 0$ (b) $k \neq 0$

(c) $-1 < k < 1$ (d) $-2 < k < 2$

Solution: The given system will have a unique solution if

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{vmatrix} \neq 0$$

$\Rightarrow\ k \neq 0.$

To be continued ...

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